POSITIVE MASS THEOREM: UNDERGRADUATE FRIENDLY MONOGRAPH

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1. Motivation

In 1979, Schoen and Yau proved that for smooth, complete asymptotically flat 3-dimensional Riemannian manifolds with non-negative scalar curvature, the total mass is non-negative [11]. Moreover, the equality holds if and only if the Riemannian manifolds is isometric to the Euclidean space [11]. The problem has been extended up to dimension 7 using an inductive argument starting from dimension 3. In 2011, Lam extended this theorem to all dimensions for graphical manifolds over the Euclidean space. This is a special class of Riemannian manifold, but the proof works in all dimensions [9]. In 2014, Dahl, Gicquaud, and Sakovich proved a similar result for n-dimensional graphical manifolds over the hyperbolic space with scalar curvature lower bound $-n\,(n-1)$ [7].

2. Introduction

I will present an undergraduate friendly monograph of the total energies and graphs involved with asymptotically flat manifolds and asymptotically hyperbolic manifolds. For this, this paper is broken into two parts: Part I is, in part, a brief review of geometric and topological terms that will be used throughout, which leads into a digest of asymptotically flat manifolds and their graphical versions. Part II utilizes the techniques from Part I but in hyperbolic space. Utilizing the divergence theorem, I will demonstrate that both asymptotically flat manifolds and asymptotically hyperbolic manifolds' total masses fall under the positive mass theorem, by demonstrating that their total energies, individually, equal a mass m. While this will give an overview of concepts, the primary focus is on the calculation aspect. For more conceptual information, I highly recommend [1] [9] as that is where I'm primarily drawing from for initial information regarding the asymptotically flat manifold and asymptotically hyperbolic manifold, respectively.

3. Preliminary

3.1. **Defining a Manifold.** A manifold is a generalized space locally linked to Euclidean space. A manifold useful for this paper is a Riemannian manifold which contains of smooth infinitesimally differential surface. I will call this manifold M and consist of the following properties:

- <u>Countable Axiom of M</u>: There exist a countable basis for the topology of M.
- <u>Hausdorff Axion</u>: Given two distinct points of M, there exist neighborhoods of these two points that do not intersect.
- <u>Locally Euclidean Space</u>: Dealing with functional compositions the connection(ϕ) between a manifold and a Euclidean Space can be in the form:continuous, bijection, and continuous inverse. Every point of p $\in$ M has a open neighborhood $\overline{U}$, and an open set $\overline{U} \subset \mathbb{R}^n$, and a homeomorphism $\phi : \overline{U} \to U$.

A manifold's properties projected onto a Euclidean space is recorded by looking at as a series of coordinate charts $\mathcal{A} = \{(U_i, \phi_i)\}$ such that covers M, i.e., $M = \bigcup_i U$. A C^∞-atlas on a topological manifold M is an atlas $\mathcal{A}$ such that for any pair i, j with $U_i \cap U_j \neq \emptyset$.The map

$$\phi_j \circ \phi_i^{-1} : \phi_i \left(U_i \cap U_j\right) \to \phi_j \left(U_i \cap U_j\right) \tag{3.1}$$

has an inverse that infinitely differentiable(i.e., has a diffeomorphism).

Definition 3.1. (Metric Tensor). A metric tensor is a symmetric nondegenerative form on a differentiable manifold.

Definition 3.2. (Lorenzian Manifold). Consider the pair (M, g) as a semi-Riemannian manifold if M is a smooth manifold and g is a non-degenerative symmetric $\begin{pmatrix}0\\2\end{pmatrix}$-tensor field on M. In local cooridnate system $\phi = \left(x^1, ..., x^n\right)$ in M, g can be expressed as

$$g = g_{ij} dx^i \otimes dx^j, \tag{3.2}$$

where $g_{ij} = g\left(\dfrac{\partial}{\partial x^i}, \dfrac{\partial}{\partial x^j}\right) \in C^\infty(M)$. At each point $p \in M$, the matrix $[g_{ij}(p)]$ of metric tensor g on M is invertible with inverse $[g^{ij}(p)]$. Another element of Lorentzian manifold is its matrix's diagonal values (+, -, -, -),i.e., the signature $sign$ of $p \in M$ is individual diagnol +,- values $(\epsilon_1, ..., \epsilon_n)$ where

$$\epsilon_i = sign\left(g_{ii}(p)\right), \tag{3.3}$$

where g_{ii} is a orthonormal basis, and the index of g at $p \in M$ is the number of negative signs in $(\epsilon_1, ...\epsilon_n)$. In the case of Lorenzian manifold, the index of g has a value of -1 (i.e., + - - -) everywhere in M. Since (2) can be written in bilinear form, the Lorentzian manifold is incredibly useful for generalizing metrics in spacetime.

Example 3.3. Euclidean Space $\mathbb{R}^n$ has a metric

$$\delta = dx_1^2 + dx_2^2 + ... + dx_n^2 \tag{3.4}$$

And it has a matrix representation

$$\delta = \begin{bmatrix} 1 & 0 & .. & \\ 0 & 1 & 0 & \ldots \\ 0 & 0 & \ddots & \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example 3.4. Schwarschild spacetime is a Lorentzian manifold with metric

$$g = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + sin^2\theta d\varphi^2\right) \tag{3.5}$$

This solution is a spherically symmetric metric which represents a black hole.

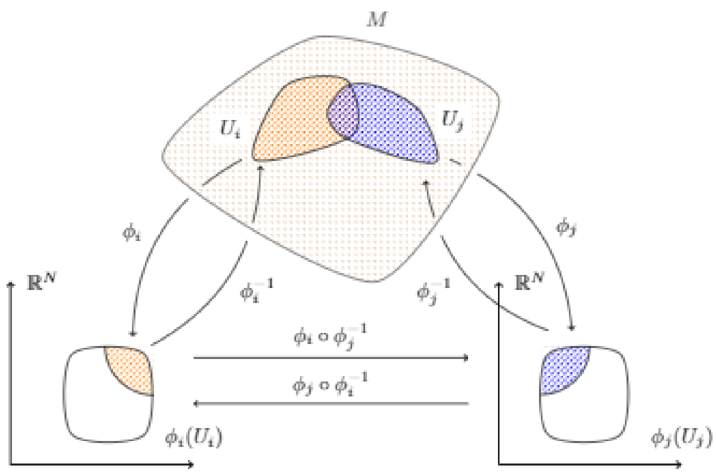


FIGURE 1. Smooth Atlas of M's Inversibility

3.2. **Curvature Tensor and Other Useful Identities.** Metrics in Euclidean space can be summarized in terms of:

- Knronecker delta (δ)

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \tag{3.6}$$

in $\mathbb{R}^n$ i, j $\in \{1, ..., n\}$.

A theorem and a series of definitions are necessary.

Definition 3.5. (Connection map ∇). A connection ∇ on a smooth manifold M is a map

$$\nabla : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$$

$$(X, Y) \to \nabla_X Y,$$

in so that f, $g \in C^\infty$(M) and X,Y,$Z \in \mathcal{X}(M)$, with $\nabla_i V$ is known as a covariant derivative with subscript i being known as the direction and V, in this instance, is known as the input field. Abstractly, the connection map/covariant derivative has the following properties in a vector space:

(1) Leibniz rule: $\nabla_X (fY) = X(f) Y + f \nabla_X Y$.(a sort of product rule)

(2) Linearity: $\nabla_X (Y+Z) = \nabla_X Y + \nabla_X Z$, (you can add then take the individual covariant derivative or vice versa)
(3) Tensorial: $\nabla_{fX+gY} Z = f\nabla_X Y + g\nabla_Y Z$, (linearity for direction vector input)

Remark 3.6. $\nabla_X Y$ is the covariant derivative of Y in the direction X for the connection ∇. To understand connection, it is also important to know that there is lie bracket. For brevity, we will say a lie bracket is the measure how much a vector field fails to close. If a lie bracket is zero then the vector field successfully closed.

Theorem 3.7 (Fundamental Theorem of Semi-Riemannian Geometry)**.** *On a semi-Riemannian manifold (M, g), there exists a unique connection ∇ such that*

(1) Torison-Free: $[X,Y] = \nabla_X Y - \nabla_Y X$, (that is to say this lie bracket is zero and that the torison tensor vectors are parallel to each other)
(2) Metric compatibility: $Xg(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$

for all $X, Y, Z \in \mathcal{X}(M)$.

Remark 3.8. $\mathcal{X}$= { all smooth vector fields on $\mathbb{R}^n$}.

Which allows for the following notation

Definition 3.9 (Christoffel Symbol)**.** Consider semi-Riemannian manifold (M, g) with coordinate frame $\{\partial_1, ..., \partial_N\}$ for a tangent space TM on an open set $U \subset M$. The expansion of $\nabla_{\partial_i}\partial_j$ in terms of coordinate frame is

$$\Gamma^k_{ij}\partial_k := \nabla_{\partial_i}\partial_j, \text{ for all } i,j,k \in \{1, ..., n\}, \tag{3.7}$$

where Γ^k_{ij} is called a Christoffel symbol of $\nabla^k_{ij}\vec{e}_k$ (where $\vec{e}_k$ is known as the eigen vector of k direction) with respect to the coordinate frame. Christoffel symbols satisfy the following thanks to the Torison-Free definition (3.7):

$$\Gamma^k_{ij} = \Gamma^k_{ji}, \text{ for all } i,j,k \in \{1,2,...,n\}, \tag{3.8}$$

$$\Gamma^k_{ij} = \frac{1}{2}g^{kl}(\partial_i g_{lj} + \partial_j g_{li} - \partial_l g_{ij}) \text{ for all } i,j,k \in \{1,2...,n\}. \tag{3.9}$$

Definition 3.10 (Ricci Curvature)**.** The Ricci curvature is a symmetric $\begin{pmatrix}0\\2\end{pmatrix}$-tensor and has a matrix representation. In local coordinate e.g., $\left(x^1, ..., x^n\right)$, the Ricci curvature is

$$Ric = R_{ij}dx^i \otimes dx^j, \tag{3.10}$$

where

$$R_{ij} = R(\partial_i, \partial_j) = \partial_k\Gamma^k_{ij} - \partial_j\Gamma^k_{ik} + \Gamma^l_{ij}\Gamma^k_{kl} - \Gamma^l_{ik}\Gamma^k_{jl}. \tag{3.11}$$

The trace of Ricci curvature is called the scalar curvature and is defined as

$$R = g^{ij}R_{ij}. \tag{3.12}$$

Since Ricci curvature is symmetric, it is diagonalizable. Therefore there exist an orthonormal frame $\{e_1, .., e_n\}$ such that

$$Ric = \begin{bmatrix} R_{11} & 0 & ... \\ 0 & R_{22} & ... \\ 0 & ..... & R_{nn} \end{bmatrix}, \tag{3.13}$$

where $R_{ij} = Ric(e_i, e_i)$. Moreover, since $g_{ij} = \delta_{ij}$, we have

$$R = R_{11} + ... + R_{nn}. \tag{3.14}$$

Definition 3.11 (Second Fundamental Form and Mean Curvature)**.** Let (N, g_m)be a $(n-1)$-dimension hypersurface of N with unit normal ν. For a semi-Riemannian hypersurface M of N, we have only one normal direction with unit normal ν and the second fundamental form in local coordinate $\{x^1, .., x^{n-1}\}$.

$$S = S_{ij} dx^i \otimes dx^j,\ S_{ij} = -g_N \left(\nabla_{\partial_i} \nu, \partial_j\right) = g_N \left(\nabla_{\partial_i} \nu, \partial_j\right). \tag{3.15}$$

Mean curvature is

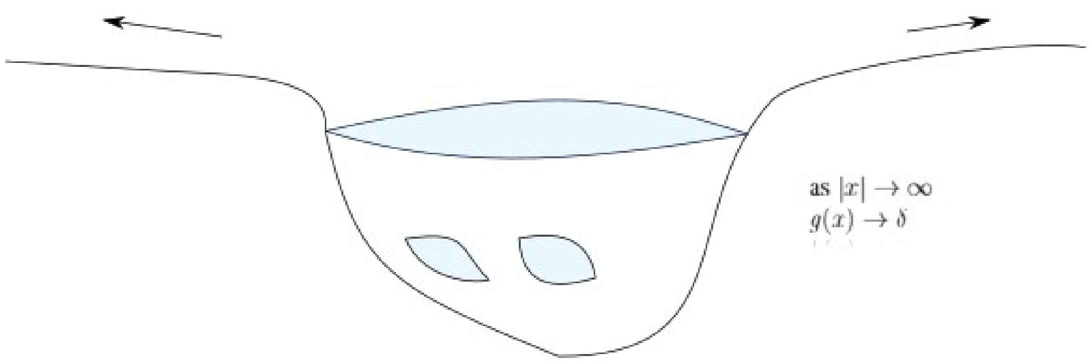
$$H = g^{ij} S_{ij}. \tag{3.16}$$

A relation between Ricci curvature, II fundamental form and mean curvature is the trace of the Gauss equation. In Euclidean space this trace is

$$R_N - Ric_N \left(\nu, \nu\right) = R_M + \left(|S|^2 - H^2\right). \tag{3.17}$$

Definition 3.12 (Graphs and Space)**.** Let M be a smooth manifold and $f : M \to \mathbb{R}$ be a smooth manifold function. A graph Σ over M is a smooth manifold of $\mathbb{R} \times M$ and defined by $\Sigma := \{(f(x), x) \in \mathbb{R} \times M\}$. If $M \leq \mathbb{R}^n$ then $M \leq \mathbb{H}^n$ for a Hyperbolic space $\mathbb{H}$.

Definition 3.13 (Positive Mass Theorem)**.** *Positive Mass Theorem* is the notion that of the initial cachy data $\left(M^3, g, k\right)$ (where M^3 is an asymptotically flat manifold, g is a riemannian metric, and k is a 2-tensor induced by a quadratic formation that takes into account how arc length changes as surfaces along a tangential direction, i.e, II fundamental form) is asymptotically flat with a surrounding trivial topology (i.e., the exterior region of an Euclidean space).



where the metric g resembles a flat metric δ and 2-tensor k goes to zero as manifold M goes to ∞

4. Asymptotically Flat (AF) Manifold

4.1. **Mass/Energy in General Relativity.** The space part of spacetime is modeled by a Riemannian manifold (M, g) where g is the metric (positive definite symmetric metric represented by a matrix).

How to defined total energy/mass of spacetime? In Newtonian physics

$$\text{energy} = \int_{\text{Domain}} \text{force}$$

Hence, what is force? One way to represent it is by the curvature of (M, g).

What is curvature of a Riemannian manifold? In 2D, we have Gauss curvature k A notion that involves k that I focused on is the scalar curvature R which is a function of g and its derivative. R can also be thought of as function $F\left(g, \partial g, \partial^2 g\right)$.

Flat $\mathbb{R}^3$ with metric δ

$$\delta = \text{identity matrix}$$

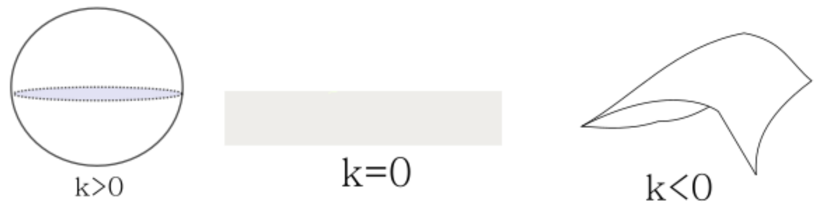


FIGURE 2. Visual responding to different k

has zero scalar curvature. Conceptually one can say

$$\text{energy} \approx \int_{\text{Domain}} \left(R - \underbrace{R_o}_{\text{curvature of model}} \right)$$

4.2. **Positive mass Theorem (PMT) for Graphs over Euclidean space.** The $m_{ADM}(M^n, g)$ of an asymptotically flat manifold (M^n, g) is defined [9] as

$$m_{ADM}(M^n, g) = \lim_{r \to \infty} \frac{1}{2(n-1)\omega_{n-1}} \times \int_{S_r} (\partial_i g_{ij} - \partial_j g_{ii}) \nu^j dS_r,$$

where ω_{n-1} is the volume of the $n-1$ unit sphere, S_r is the coordinate sphere of radius r. ν is the outward unit normal to S_r and dS_r is the area element of S_r in coordinate chart [9]. Another way of writing the ADM mass of a graph() can be written as

$$m_{ADM} = \lim_{r \to \infty} \frac{1}{16\pi} \int_{S_r} \frac{1}{1 + |\nabla f|^2} (f_{ii} f_j - f_{ij} f_i) \nu^j dA, \tag{4.1}$$

Which leads us into the theorem

Theorem 4.1. *If (M^3, g) is a complete, asymptotically flat riemanninan 3-manifold with nonnegative scalar curvature and ADM mass m_{ADM}, then*

$$m_{ADM} \geq 0,$$

with $m_{ADM} = 0$ if and only if (M^3, g) is isometric to $\mathbb{R}^3$ with the standard flat metric.

and with this paper, can generalize the PMT to be

$$m_{ADM} \geq \int_{\mathbb{R}^n} \frac{R_g}{\sqrt{1 + |\nabla f|^2}} dVol_g. \tag{4.2}$$

For here, I will demonstrate this is true by showing $m_{ADM} = m$ and, with the divergence theorem, derive $\int_{\mathbb{R}^n} \frac{R_g}{\sqrt{1+|\nabla f|^2}} dVol_g$ and why it equals zero.

4.3. **Proof of PMT for graphs over Euclidean space.**

$$
\begin{aligned}
m_{ADM}(g) &= m_{ADM}(\delta) - \lim_{r\to\infty}\frac{1}{8\pi}\int_{S_r}\frac{\partial}{\partial_r}\left(1+\frac{m}{2r}\right)^4 dS_r \\
&= 0 - \lim_{r\to\infty}\frac{1}{8\pi}\int_{S_r}4\left(1+\frac{m}{2r}\right)^3\left(-\frac{m}{2r^2}\right)dS_r \\
&= -\lim_{r\to\infty}\frac{1}{8\pi}4\pi r^2 4\left(1+\frac{m}{2r}\right)^3\left(-\frac{m}{2r^2}\right) \\
&= -\lim_{r\to\infty}\frac{1}{8\pi}4\pi r^2\left(-\frac{4m^4}{16r^5}-\frac{12m^3}{8r^4}-\frac{12m^2}{4r^3}-\frac{4m}{2r^2}\right) \\
&= -\lim_{r\to\infty}\frac{r^2}{2}\left(-\frac{4m^4}{16r^5}-\frac{12m^3}{8r^4}-\frac{12m^2}{4r^3}-\frac{4m}{2r^2}\right) \\
&= -\lim_{r\to\infty}\left(-\frac{4m^4}{32r^3}-\frac{12m^3}{16r^2}-\frac{12m^2}{8r}-\frac{4m}{4}\right) \\
&= -\frac{-4m}{4} \\
&= m.
\end{aligned}
\tag{4.3}
$$

Remark 4.2. δ is a standard metric of which a complete asymptotically flat manifold is simply the Euclidean space $\mathbb{R}^n$. Since $\partial_k\delta_{ij}=0$ for all i,j and k, $m_{ADM}(\delta)$ is 0.

4.3.1. *PullBack Example with Smooth Map F.* Now for $\int_{\mathbb{R}^n}\frac{R_g}{\sqrt{1+|\nabla f|^2}}dVol_g$ we will refer to section 3.1 in [1] using pullback method to induce metric g, which displays the following lemma:

Lemma 4.3. *For smooth function $f:\mathbb{R}^n\to\mathbb{R}$, the graph of f is a hypersurface in $\mathbb{R}^{n+1}$. By letting manifold M*

$$M^n=\{(x_1,...,x_n,f(x_1,...,x_n))\in\mathbb{R}^{n+1}:(x_1,...,x_n)\in\mathbb{R}^n\} \tag{4.4}$$

be the graph of f, then (M^n,g) is isometric to $(\mathbb{R}^n,\delta+df\otimes df)$. If we say F is a smooth map where

$$
\begin{aligned}
F:(\mathbb{R}^n,\delta+df\otimes df)&\to(M^n,g)\\
x&\to(x,f(x))
\end{aligned}
\tag{4.5}
$$

*then F is a diffeomorphism whose smooth inverse is the projection map π: $(M^n,g)\to(\mathbb{R}^n,\nabla+df\otimes df)$ defined by $\pi(x,f(x))=x$. So for pullback denoted by *, we now claim:*

$$F_*\frac{\partial}{\partial x^i}=\frac{\partial}{\partial x^i}+f_i\frac{\partial}{\partial x^{n+1}}, \tag{4.6}$$

where shorthand notation $f_i=\dfrac{\partial f}{\partial x^i}$.

Here is my detailed calculation of this claim

Proof. For all $i, j \in \{1, ..., n\}$, if $\phi \in C^{\infty}(M^n, g)$, then

$$
\begin{aligned}
\left(F\frac{\partial}{\partial x^i}\right)\phi &= \frac{\partial}{\partial x^i}(\phi \circ F) \\
&= \frac{\partial}{\partial x^i}(\phi(x, f(x))) \\
&= \left(\sum_{k=1}^{n+1} \frac{\partial \phi}{\partial x^k}\frac{\partial x^k}{\partial x^i}\right) \\
&= \frac{\partial \phi}{\partial x^{n+1}}\frac{\partial f}{\partial x^i} + \left(\sum_{k=1}^{n} \frac{\partial \phi}{\partial x^k}\frac{\partial x^k}{\partial x^i}\right) \qquad (4.7) \\
&= \frac{\partial \phi}{\partial x^{n+1}}\frac{\partial f}{\partial x^i} + \left(\frac{\partial \phi}{\partial x^i}\frac{\partial x^i}{\partial x^i}\right) \\
&= \frac{\partial \phi}{\partial x^{n+1}}\frac{\partial f}{\partial x^i} + \left(\frac{\partial \phi}{\partial x^i}\right) \\
&= \left(\frac{\partial}{\partial x^i} + \frac{\partial f}{\partial x^i}\frac{\partial}{\partial x^{n+1}}\right)\phi,
\end{aligned}
$$

therefore $F_*\frac{\partial}{\partial x^i} = \frac{\partial}{\partial x^i} + f_i\frac{\partial}{\partial x^{n+1}}$. □

Remark 4.4. Note: partial derivative $\frac{\partial}{\partial \overline{x}^i}$ can be written in the form

$$\frac{\partial}{\partial \overline{x}^i} = \frac{\partial x^j}{\partial \overline{x}^i}\frac{\partial}{\partial x^j}. \qquad (4.8)$$

4.3.2. *PullBack in relation to Induced Metric g.* Knowing g is an induced metric of $\mathbb{R}^{n+1}$,

$$g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \delta_{ij} \ for \ 1 \leq i, j \leq n+1, \qquad (4.9)$$

so for $g\left(F\frac{\partial}{\partial x^i}, F\frac{\partial}{\partial x^j}\right)$,

$$
\begin{aligned}
g\left(F\frac{\partial}{\partial x^i}, F\frac{\partial}{\partial x^j}\right) &= g\left(\frac{\partial}{\partial x^i} + f_i\frac{\partial}{\partial x^{x+1}}, \frac{\partial}{\partial x^j} + f_j\frac{\partial}{\partial x^{n+1}}\right) \qquad (4.10) \\
&= \delta_{ij} + f_i f_j.
\end{aligned}
$$

To understand the relation between scalar curvature R and smooth map F, we need to understand mixed II fundamental form and how that leads to a graphing of the scalar curvature R.

4.3.3. *Mixed 1st Fundamental Form.* To deal with II fundamental form change indices, cancel matching upper and lower indices for each changed form as shown:

$$\delta_i^k = \delta_{ij} g^{jk}, \qquad (4.11)$$

Of which (4.11)) is a demonstrative example of canceling Christoffel symbols. Also note:

$$g_{ij} = \delta_{ij} + f_i f_j, \qquad (4.12)$$

$$g^{ij} = \delta^{ij} - \frac{f^i f^j}{1 + |\nabla f|^2}. \qquad (4.13)$$

For proof of (4.13)

Proof. Let $f \in C^\infty(M^n)$ where $\left(\tilde{M}^n, \tilde{g}\right) = (M^n, g_{ij} + f_i f_j)$ with g_{ij} being δ_{ij} in a flat asymptotic surface. where we known the Einstein summation notation $g = g_{ij} + f_i f_j$.

$$g = [\delta_{ij} + g_{ij}] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} f_i^2 & f_i f_j \\ f_{ij} & f_j^2 \end{bmatrix} = \begin{bmatrix} 1 + f_i^2 & f_i f_j \\ f_{ij} & 1 + f_j^2 \end{bmatrix}$$

for det g

$$\begin{aligned} \det \text{g} &= \left(1 + f_i^2\right)\left(1 + f_j^2\right) - (f_i f_j)^2 \\ &= 1 + f_j^2 + f_i^2 + f_i^2 f_j^2 - f_i^2 f_j^2 \\ &= 1 + f_j^2 + f_i^2 \\ &= 1 + |\nabla f| \end{aligned}$$

thus

$$g^{ij} = \frac{1}{1 + |\nabla f|^2} \begin{bmatrix} 1 + f_i^2 & -f_i f_j \\ -f_{ij} & 1 + f_j^2 \end{bmatrix}$$

in other words

$$g^{ij} = \delta^{ij} - \frac{f^i f^j}{1 + |\nabla f|^2}$$

□

Now that I have established (4.13) as the inverse matrix, I use (4.7) and cancellation method (4.11), to find the product of (4.12) and (4.13) to be

$$\begin{aligned} g_{ij} g^{jk} &= (\delta_{ij} + f_i f_j)\left(\delta^{jk} - \frac{f^j f^k}{1 + |\nabla f|^2}\right) \\ &= \left(\delta_{ij}\delta^{jk} - \frac{\delta_{ij} f^j f^k}{1 + |\delta f}^2 | + f_i f_j \delta^{jk} - \frac{f_i f_j f^j f^k}{1 + |\nabla f}^2 |\right) \\ &= \delta_i^k - \frac{f_i f^k}{1 + |\nabla f|^2} + f_i f^k - \frac{f_i f^k |\nabla f|^2}{1 + |\nabla f|^2} \qquad (4.14) \\ &= \delta_i^k + f_i f^k - \frac{f_i f^k \left(1 + |\nabla f|^2\right)}{1 + |\nabla f|^2} \\ &= \delta_i^k + f_i f^k - f_i f^k \\ &= \delta_i^k. \end{aligned}$$

The partial derivative ∂_k of (4.12) is

$$\partial_k g_{ij} = \delta_k (\delta_{ij} + f_i f_j) = f_{ik} f_j + f_i f_{jk}. \qquad (4.15)$$

Going off (4.15) as an example, the Christoffel symbol Γ_{ij}^k is

$$
\begin{aligned}
\Gamma_{ij}^k &= \frac{1}{2} g^{km} \left(\partial_j g_{im} + \partial_i g_{jm} - \partial_m g_{ij}\right) \\
&= \frac{1}{2}\left(\delta^{km} - \frac{f^k f^m}{1+|\nabla f|^2|^2}\right)\left(f_{ij} f_m + f_i f_{mj} + f_{ji} f_m\right. \\
&\left. + f_j f_{mi} - f_{im} f_j - f_i f_{jm}\right) \\
&= \frac{1}{2}\left(\delta^{km} - \frac{f^k f^m}{1+|\nabla f|^2}\right)\left(2 f_{ij} f_m\right) \\
&= \left(\delta^{km} - \frac{f^k f^m}{1+|\nabla f|^2}\right)\left(f_{ij} f_m\right) \\
&= \delta^{km}\left(f_{ij} f_m\right) - \frac{f^k f^m \left(f_{ij} f_m\right)}{1+|\nabla f|^2} \\
&= f^k f_{ij} - \frac{\left(f^k f_{ij} |\nabla f|^2\right)}{1+|\nabla f|^2} \\
&= f^k f_{ij}\left(1 - \frac{|\nabla f|^2}{1+|\nabla f|^2}\right) \\
&= f^k f_{ij}\left(\frac{1+|\nabla f|^2 - |\nabla f|^2}{1+|\nabla f|^2}\right) \\
&= f^k f_{ij}\left(\frac{1}{1+|\nabla f|^2}\right) \\
&= \left(\frac{f^k f_{ij}}{1+|\nabla f|^2}\right).
\end{aligned} \tag{4.16}
$$

Remark 4.5. Observe that

$$f^m f_m = |\nabla f|^2. \tag{4.17}$$

Furthermore

$$|\nabla f|^2 = f_l f_l. \tag{4.18}$$

So for the Christoffel symbol in (48) taking the partial derivative ∂_k is:

$$\partial_k \Gamma_{ij}^k = \frac{f_k f_{ijk}}{1+|\nabla f|^2} + \frac{f_{kk} f_{ij}}{1+|\nabla f|^2} - \frac{2 f_{ij} f_{kl} f_k f_l}{\left(1+|\nabla f|^2\right)^2}. \tag{4.19}$$

Proof:

$$
\begin{aligned}
\partial_k \Gamma_{ij}^k &= \partial_k \left(\frac{f^k f_{ij}}{1+|\nabla f|^2}\right) \\
&= \left(\frac{f^k f_{ijk}}{1+|\nabla f|^2}\right) + \partial_k \left(\frac{f^k f_{ijk}}{1+|\nabla f|^2}\right) \\
&= \left(\frac{f^k f_{ijk}}{1+|\nabla f|^2}\right) + \left(\frac{f^{kk} f_{ij}}{1+|\nabla f|^2}\right) + \partial_k \left(\frac{f^k f_{ij}}{1+|\nabla f|^2}\right) \\
&= \left(\frac{f^k f_{ijk}}{1+|\nabla f|^2} + (\frac{f^{kk} f_{ij}}{1+|\nabla f|^2}\right) - 2\left(f_{kl} f_l\right) \frac{f_{ij} f_k}{\left(1+|\nabla f|^2\right)^2} \\
&= \frac{f_k f_{ijk}}{1+|\nabla f|^2} + \frac{f_{kk} f_{ij}}{1+|\nabla f|^2} - \frac{2 f_{ij} f_{kl} f_k f_l}{\left(1+|\nabla f|^2\right)^2}.
\end{aligned} \tag{4.20}
$$

Now we can effectively relate Christoffel symbols to the scalar curvature. For example,

$$\begin{aligned}
\partial_k \Gamma^k_{ij} &= \frac{f_{ij\partial k} f_k}{1+|\nabla f|^2} + \frac{f_{ij} f_{kk}}{1+|\nabla f|^2} - \frac{2 f_{ij} f_{kl} f_k f_l}{\left(1+|\nabla f|^2\right)^2}, \\
\partial_j \Gamma^k_{ik} &= \frac{f_{ijk} f_k}{1+|\nabla f|^2} + \frac{f_{ik} f_{jk}}{1+|\nabla f|^2} - \frac{2 f_{ik} f_{jl} f_k f_l}{\left(1+|\nabla f|^2\right)^2}, \\
\Gamma^l_{ij}\Gamma^k_{kl} &= \frac{f_{ij} f_{kl} f_k f_l}{\left(1+|\nabla f|^2\right)^2}, \\
\Gamma^l_{ik}\Gamma^k_{jl} &= \frac{f_{ik} f_{jl} f_k f_l}{\left(1+|\nabla f|^2\right)^2}.
\end{aligned} \tag{4.21}$$

Thus, using the definition of scalar curvature (3.12) we have

(4.22)

$$\begin{aligned}
R &= g^{ij}\left(\Gamma^k_{ijk} - \Gamma^k_{ikj} + \Gamma^l_{ij}\Gamma^k_{kl} - \Gamma^l_{ik}\Gamma^k_{jl}\right) \\
&= \left(\delta_{ij} - \frac{f_i f_j}{1+|\nabla f|^2}\right)\left(\cancel{\frac{f_{ijk} f_k}{1+|\nabla f|^2}} + \frac{f_{ij} f_{kk}}{1+|\nabla f|^2} - \cancel{\frac{2 f_{ij} f_{kl} f_k f_l}{(1+|\nabla f|^2)^2}}\right. \\
&\quad \left. - \cancel{\frac{f_{ijk} f_k}{1+|\nabla f|^2}} - \frac{f_{ik} f_{jk}}{1+|\nabla f|^2} + \cancel{\frac{2 f_{ik} f_{jl} f_k f_l}{(1+|\nabla f|^2)^2}} + \frac{f_{ij} f_{kl} f_k f_l}{(1+|\nabla f|^2)^2} - \frac{f_{ik} f_{jl} f_k f_l}{(1+|\nabla f|^2)^2}\right) \\
&= \left(\delta_{ij} - \frac{f_i f_j}{1+|\nabla f|^2}\right)\left(\frac{f_{ij} f_{kk}}{1+|\nabla f|^2} - \frac{f_{ik} f_{jk}}{1+|\nabla f|^2} + \frac{f_{ij} f_{kl} f_k f_l}{\left(1+|\nabla f|^2\right)^2} - \frac{f_{ik} f_{jl} f_k f_l}{\left(1+|\nabla f|^2\right)^2}\right) \\
&= \underbrace{\frac{1}{1+|\nabla f|^2}\left(f_{ii} f_{kk} - f_{ik} f_{ik}\right) - \frac{f_k f_l}{\left(1+|\nabla f|^2\right)^2}\left(f_{ii} f_{kl} - f_{ik} f_{il}\right)}_{\text{distributed } \delta_{ij}} - \\
&\quad \underbrace{\frac{f_i f_j}{\left(1+|\nabla f|^2\right)^2}\left(f_{ij} f_{kk} - f_{ik} f_{jk}\right) - \frac{f_i f_j f_k f_l}{\left(1+|\nabla f|^2\right)^3}\left(f_{ij} f_{kl} - f_{ik} f_{il}\right)}_{\text{distributed } -\frac{f_i f_j}{1+|\nabla f|^2}} \\
&= \frac{1}{1+|\nabla f|^2}\left(f_{ii} f_{kk} - f_{ik} f_{ik}\right) - \frac{f_k f_l}{\left(1+|\nabla f|^2\right)^2}\left(f_{ii} f_{kl} - f_{ik} f_{il}\right) - \frac{f_i f_j}{\left(1+|\nabla f|^2\right)^2}\left(f_{ij} f_{kk} - f_{ik} f_{jk}\right).
\end{aligned}$$

For which

$$\frac{1}{(1+|\nabla f|^2|)} f_{ii} f_{kk} - f_{ik} f_{ik} \to \frac{1}{(1+|\nabla f|^2|)} f_{ii} f_{jj} - f_{ij} f_{ij}, \tag{4.23}$$

$$-\frac{f_k f_l}{(1+|\nabla f|^2|)^2} f_{ii} f_{kl} - f_{ik} f_{il} \to -\frac{f_k f_j}{(1+|\nabla f|^2|)^2} f_{ii} f_{kj} - f_{ik} f_{ij}, \tag{4.24}$$

$$-\frac{f_i f_j}{(1+|\nabla f|^2|)^2} f_{ij} f_{kk} - f_{ik} f_{jk} \to -\frac{f_k f_j}{(1+|\nabla f|^2|)^2} f_{kj} f_{ii} - f_{ki} f_{ji}, \tag{4.25}$$

where (55) consists of the first term's k indices changed to j. (56) consists of the second term's l indices changed to j. (57)consists of the second term's i indices changed to k. All together equals the scalar curvature R of a graph $(\mathbb{R}^n, \delta + df \otimes df)$ is

$$R = \frac{1}{1+|\nabla f|^2}\left(f_{ii} f_{jj} - f_{ij} f_{ij} - \frac{2 f_j f_k}{1+|\nabla f|^2}\left(f_{ii} f_{jk} - f_{ij} f_{ik}\right)\right). \tag{4.26}$$

with a visual of In addition, with the given properties:

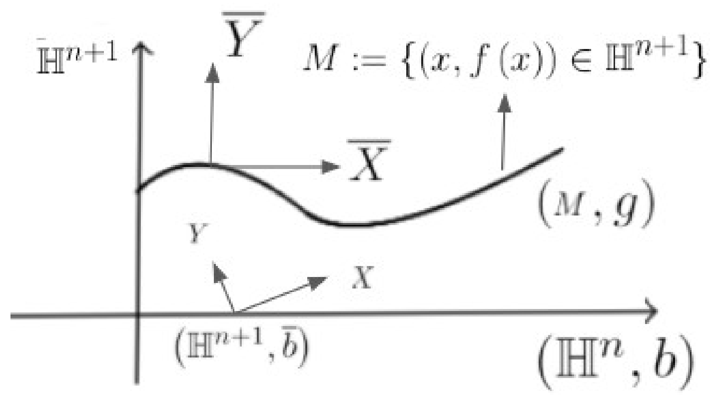


Figure 3. Visual of II graph embedded in Euclidean space

$$f_{ii}f_{jj} = (\nabla f)^2, \tag{4.27}$$
$$f_{ij}f_{ij} = ||H^J||^2, \tag{4.28}$$
$$f_{ii}f_{jk}f_jf_k = (\nabla f)\, H^f(\nabla f, \nabla f), \tag{4.29}$$
$$f_{ij}f_{ik}f_jf_k = ||H^f(\nabla f, .)||^2, \tag{4.30}$$
where $H^f(\nabla f, .)$ from (62) is the 1-form and takes a vector ν to $H^f(\nabla, \nu)$. Also, given (59) – (62) it is clear how scalar curvature of a graph has the coordinate-free expression
$$R = \frac{1}{1+|\nabla f|^2}\left((\nabla f)^2 - ||H^f||^2 - \frac{2\nabla f H^2(\nabla f, \nabla f) + 2||H^f(\nabla f, \cdot)||^2}{1+|\nabla f|^2}\right). \tag{4.31}$$

Lemma 4.6. *The scalar curvature can be rewritten as*
$$R = \nabla \cdot \left(\frac{1}{1+|\nabla f|^2}\left(f_{ii}f_j - f_{ij}f_i\right)\partial_j\right).$$

Proof.
$$\begin{aligned} R =& \frac{1}{1+|\nabla f|^2}\left(f_{iij}f_j - f_{ij}f_{ij} - \frac{2f_jf_k}{1+|\nabla f|^2}\left(f_{ii}f_{jk} - f_{ij}f_{ik}\right)\right), \\ =& \nabla \cdot \left(\frac{1}{1+|\nabla f|^2}\left(f_{ii}f_j - f_{ij}f_i\right)\partial_j\right). \end{aligned}$$
□

Lemma 4.7. *Divergence Theorem Let (M, g) be a Riemannian manifold with boundary ∂M. Let $div_g =: TM \to \mathbb{R}$ be the divergence operator ,i.e., $div_g x = \nabla_i x^i$ for vector field $X = X^i\dfrac{\partial}{\partial X^i} \in TM$. Then $\int_M div_g X\, dVol_g = \int_{\partial M} X \cdot \nu dA$ where ν is the unit normal vector on ∂M*

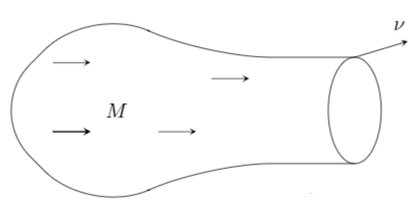


Figure 4. Riemannian manifold with boundary ∂M

Proof of Theorem **??**. By definition, the ADM mass of $(M^n, g) = (\mathbb{R}^n, \delta + df \otimes df)$ is (4.1)

$$m_{ADM} = \lim_{r\to\infty} \frac{1}{2(n-1)\omega_{n-1}} \int_{S_r} \frac{1}{1+|\nabla f|^2} (f_{ii}f_j - f_{ij}f_i)\,\nu_j dS_r.$$

Now applying the divergence theorem in $(\mathbb{R}^n, \delta)$ and use Lemma (4.6) to get

$$m_{ADM} = \frac{1}{2(n-1)\omega_{n-1}} \int_{\mathbb{R}^n} \nabla \cdot \left(\frac{1}{1+|\nabla f|^2} (f_{ii}f_j - f_{ij}f_i)\,\partial_j \right) dVol_\delta,$$

since $dVol_g = \sqrt{\det g}\,dVol_\delta = \sqrt{1+|\nabla f|^2}dVol_\delta$

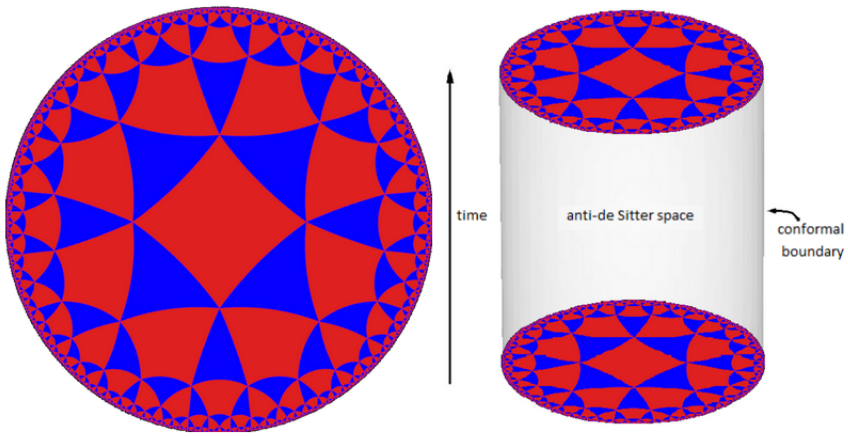
$$\begin{aligned} m_{ADM} =& \frac{1}{2(n-1)\omega_{n-1}} \int_{\mathbb{R}^n} \mathrm{R}\ dVol_\delta, \\ =& \int_{\mathbb{R}^n} \frac{R_g}{\sqrt{1+|\nabla f|^2}} dVol_g \end{aligned}$$

thus we can conclude (4.2) is true, and the PMT is valid. □

PART II

5. Asymptotically Hyperbolic (AH) Manifold

5.1. Hyperbolic Space via AdS/CFT Correspondence.



wiki/AdS/CFTcorrespondence

Anti-de-Sitter(AdS) relates to a hyperbolic spacetime. The related spacetime's energy can be generalized with a negative cosmological constant Λ. Conformal Field Theory(CFT) relates to studying certain symmetries (hence "conformal") that are involved in studying the forces of nature (hence field theory). Ads/CFT correspondence is a useful property that allows for previous calculations in quantum field theory to be solved using works from general relativity and vice versa. The picture above is a visual demonstration of the distance between points in hyperbolic space. Notice that squares and triangles are bent: in Euclidean space they would not be, and this is primarily due to a negative value for Λ. Dimensions in hyperbolic spaces can be viewed as "stacked" on top of each other.
With this in mind, the graph as seen (1) of an asymptotically hyperbolic manifold is

$$\Sigma := \{(x, s) \in \mathbb{H}^n \times \mathbb{R} | f(x) = s\} = F^{-1}(0)$$

with decay property $g = b + \underbrace{O\left(r^{-(n-1)}\right)}_{\text{decay}}$

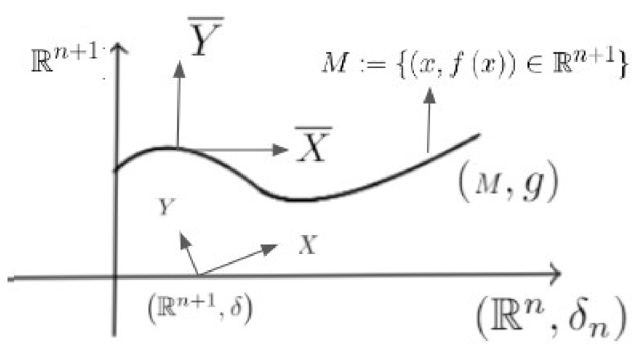


Figure 5. Graph of Hyperbolic graph

5.2. **Total mass of AH Manifolds.** As shown here[] the metric of hyperbolic space is $b = \dfrac{dr^2}{1+r^2} + r^2 g_{s^2}$. The total of (M, g) is defined by

$$mass_{AH}(g) = \lim_{r\to\infty} \int_{S_r} \left(\sqrt{1+r^2} \left(div^b e - dtr^b e \right) + \left(tr^b e \right) d\sqrt{1+r^2} - e \left(\nabla^b \sqrt{1+r^2}, \cdot \right) \right) (\nu_r)\, dS_r.$$

for which $mass_{AH} = m$.

Proof. To compartmentalize $mass_{AH}$

$$mass_{AH}(g) = \lim_{r\to\infty} \frac{1}{16\pi} \int_{S_r} \left(\sqrt{1+r^2} \left(div^b e - \mathrm{d}\ tr^b e \right) + \left(tr^b e \right) d\sqrt{1+r^2} - e \left(\nabla^b \sqrt{1+r^2}, \cdot \right) \right) (\nu_r)\, dS_r.$$

I will label the following M, W, X, Z, B, and show that only M remains

$$mass_{AH}(g) = \lim_{r\to\infty} \frac{1}{16\pi} \int_{S_r} \Bigg(\underbrace{\sqrt{1+r^2}\ div^b e\,(\nu_r)}_{M} - \underbrace{\sqrt{1+r^2}\ \mathrm{d}\ tr^b e\,(\nu_r)}_{W} + \underbrace{tr^b e \cdot \nu_r\ d\sqrt{1+r^2}}_{X}.$$

$$+ \underbrace{\nu_r tr^b e\ d\sqrt{1+r^2}}_{Z} - \underbrace{e\left(\nabla^b \sqrt{1+r^2}, \nu_r\right)}_{B} \Bigg)\, dS_r.$$

To start, we will calculate the chirstoeffel symbols

$$\begin{aligned} \Gamma^r_{rr} & \\ &= \frac{1}{2} b^{rm} \left(\partial_r b_{rm} + \partial_r b_{rm} - \partial_m b_{rr} \right), \\ &= \frac{1}{2} b^{rr} \left(\partial_r b_{rr} + \partial_r b_{rr} - \partial_r b_{rr} \right), \\ &= \frac{1}{2} \left(1+r^2\right) \left(\partial_r \left(\frac{1}{1+r^2} \right) \right), \\ &= \frac{1}{2} \left(1+r^2\right) \left(\frac{-1}{\left(1+r^2\right)^2} \cdot (2r) \right), \\ &= \frac{-r}{1+r^2}. \end{aligned} \tag{5.1}$$

(5.2)
$$
\begin{aligned}
\Gamma^r_{\theta\theta} =& \frac{1}{2} b^{rm} \left(\partial_{\theta\theta} b_{rm} + \partial_\theta b_{\theta m} - \partial_m b_{\theta\theta} \right), \\
=& \frac{1}{2} \underbrace{b^{r\theta}}_{0} \left(\partial_\theta b_{\theta\theta} + \partial_\theta b_{\theta\theta} - \partial_r b_{rr} \right), \\
&+ \frac{1}{2} b^{rr} \left(\underbrace{\partial_\theta b_{\theta r}}_{0} + \underbrace{\partial_\theta b_{\theta r}}_{0} - \partial_r b_{\theta\theta} \right), \\
=& \frac{-1}{2} b^{rr} \left(\partial_r b_{\theta\theta} \right), \\
=& \frac{-1}{2} \left(1 + r^2\right) (2r), \\
=& - \left(r + r^3\right). \\
\Gamma^r_{\phi\phi} =& \frac{1}{2} b^{rm} \left(\partial_\phi b_{\phi m} + \partial_\phi b_{\phi m} - \partial_m b_{\phi\phi} \right), \\
=& \frac{1}{2} \underbrace{b^{r\phi}}_{0} \left(\partial_\phi b_\phi + \partial_\phi b_\phi - \partial_\phi b_{\phi\phi} \right), \\
&+ \frac{1}{2} b^{rr} \left(\underbrace{\partial_\phi b_{\phi r}}_{0} + \underbrace{\partial_\phi b_{\phi r}}_{0} - \partial_r b_{\phi\phi} \right), \\
=& \frac{-1}{2} b^{rr} \left(\partial_r b_{\phi\phi} \right), \\
=& \frac{-1}{2} \left(1 + r^2\right) \left(2r \sin^2\theta\right), \\
=& - \left(1 + r^2\right) \left(2r \sin^2\theta\right).
\end{aligned}
$$

Before calculating the individual components, let us first calculate key variables that will be used throughout to avoid repetitive display

(5.3)
$$
\begin{aligned}
b^{rr} &= \left(1 + r^2\right). \\
e_{rr} &= \left(\frac{1}{1 + r^2 - \frac{2m}{r}} - \frac{1}{1 + r^2} \right) \approx \frac{-2m}{r^5} + O\left(r^{-6}\right). \\
\partial_r e_{rr} &= \left(\frac{-1}{\left(1 + r^2 - \frac{2m}{r}\right)^2} \cdot \left(2r + \frac{2m}{r^2}\right) + \frac{1}{\left(1 + r^2\right)^2} \cdot (2r) \right) \approx \frac{10m}{r^6} + \underbrace{\tilde{O}\left(r^{-6}\right)}_{\text{some variation of } O(r^{-6})}. \\
b^{rr} \partial_r e_{rr} &= \left(\frac{-\left(2r + \frac{2m}{r^2}\right)\left(1 + r^2\right)}{\left(1 + r^2 - \frac{2m}{r}\right)^2} + \frac{(2r)}{\left(1 + r^2\right)} \right).
\end{aligned}
$$

As for components from the labels M, W, X, Z, B, we will make the final calculations easier to read. Starting with X and Z's $\left(d\sqrt{1+r^2}\right)\nu_r$

$$
\begin{aligned}
\left(d\sqrt{1+r^2}\right)\nu_r =& \nu_r\left(\sqrt{1+r^2}\right), \\
=& \underbrace{\sqrt{1+r^2}\partial_r}_{\nu_r}\left(\sqrt{1+r^2}\right), \\
=& \sqrt{1+r^2}\frac{1}{2}\left(1+r^2\right)^{-\frac{1}{2}}2r, \\
=& r.
\end{aligned}
\tag{5.4}
$$

As for part of label B

$$
\begin{aligned}
e\left(\nabla^b\sqrt{1+r^2}, \nu_r\right) =& e_{rr}b^{rr}\left(\partial_r\sqrt{1+r^2}\right)\left(\sqrt{1+r^2}\right), \\
=& e_{rr}\left(1+r^2\right)\left(\sqrt{1+r^2}\right)\left(\partial_r\sqrt{1+r^2}\right), \\
=& \left(1+r^2\right)\left(\frac{1}{1+r^2-\frac{2m}{r}} - \frac{1}{1+r^2}\right)\left(\sqrt{1+r^2}\right)\left(\frac{1}{2}\left(1+r^2\right)^{-\frac{1}{2}}2r\right), \\
=& \left(\frac{1}{1+r^2-\frac{2m}{r}} - \frac{1}{1+r^2}\right)(r)\left(1+r^2\right).
\end{aligned}
\tag{5.5}
$$

Now for part of X and Z

$$
\begin{aligned}
tr^b\ e =& b^{ij}e_{ij}, \\
=& b^{rr}e_{rr}, \\
=& \left(1+r^2\right)\left(\frac{1}{1+r^2-\frac{2m}{r}} - \frac{1}{1+r^2}\right), \\
=& \frac{1+r^2}{1+r^2-\frac{2m}{r}} - 1.
\end{aligned}
\tag{5.6}
$$

Now for a part of W

$$
\begin{aligned}
\mathrm{d}\ tr_b\ e\left(\nu_r\right) =& \nu_r tr_b e, \\
=& \sqrt{1+r^2}\partial_r\left(\frac{1+r^2}{1+r^2-\frac{2m}{r}} - 1\right), \\
=& \sqrt{1+r^2}\left(\frac{2r\left(1+r^2-\frac{2m}{r}\right) - \left(1+r^2\right)\left(2r+\frac{2m}{r^2}\right)}{\left(1+r^2-\frac{2m}{r}\right)^2}\right).
\end{aligned}
$$

With those out of the way, see that with label X+label B -using (5.4), (5.6), (5.5)- that they equal zero

$$tr^b e \cdot \left(d\sqrt{1+r^2}\right)\nu_r - e\left(\nabla^b\sqrt{1+r^2},\nu_r\right) = \left(\frac{\left(1+r^2\right)}{1+r^2-\dfrac{2m}{r}}-1\right)(r) - \left(\frac{\left(1+r^2\right)}{1+r^2-\dfrac{2m}{r}}-1\right)(r).$$
$$=0$$

Also see that with (5.4), (5.6) that label W+ label Z equal zero

$$\begin{aligned}
-d\sqrt{1+r^2}\nu_r tr^b e + \sqrt{1+r^2}\nu_r d\; tr^b e = & -d\sqrt{1+r^2}\nu_r tr^b e + \sqrt{1+r^2}\nu_r tr^b e,\\
= & -\left(1+r^2\right)\left(\frac{\left(1+r^2-\dfrac{2m}{r}\right)(2r)-\left(1+r^2\right)\left(2r+\dfrac{2m}{r}\right)}{\left(1+r^2-\dfrac{2m}{r}\right)^2}\right),\\
& +\sqrt{1+r^2}\left(\frac{\left(1+r^2-\dfrac{2m}{r}\right)(2r)-\left(1+r^2\right)\left(2r+\dfrac{2m}{r}\right)}{\left(1+r^2-\dfrac{2m}{r}\right)^2}\right),\\
= & 0.
\end{aligned}$$

Now for label M see that we get the final result of m for $mass_{AH}$. Let us break this down to the components

$$div^b e\left(\nu^r\right) = \sqrt{1+r^2}div^b\left(\partial_r\right) = \sqrt{1+r^2}\left(\underbrace{b^{rr}\partial_r e_{rr}}_{G} - \underbrace{\left(b^{rr}\Gamma^r_{rr}+b^{\theta\theta}\Gamma^r_{\theta\theta}+b^{\phi\phi}\Gamma^r_{\phi\phi}\right)e_{rr}}_{M} - \underbrace{b^{rr}\Gamma^r_{rr}e_{rr}}_{Y}\right).$$

Distributing out M with the outer components

$$\frac{1}{16\pi}\int_{S_r}\sqrt{1+r^2}\sqrt{1+r^2}\left(b^{rr}\Gamma^r_{rr}+b^{\theta\theta}\Gamma^r_{\theta\theta}+b^{\phi\phi}\Gamma^r_{\phi\phi}\right)e_{rr}dS_r = \frac{1}{16\pi}\int_{S_r}\left(1+r^2\right)\left(b^{rr}\Gamma^r_{rr}+b^{\theta\theta}\Gamma^r_{\theta\theta}+b^{\phi\phi}\Gamma^r_{\phi\phi}\right)e_{rr}dS_r,$$

leaving out $\dfrac{1}{2}$ and substituting in (5.1) and (5.2)

$$\begin{aligned}
&\frac{1}{8\pi}\int_{S_r}\left(1+r^2\right)\left(b^{rr}\Gamma^r_{rr}+b^{\theta\theta}\Gamma^r_{\theta\theta}+b^{\phi\phi}\Gamma^r_{\phi\phi}\right)e_{rr}dS_r,\\
&=\frac{1}{8\pi}\left(1+r^2\right)\left(\underbrace{\left(1+r^2\right)\left(-\frac{r}{1+r^2}\right)}_{b^{rr}\Gamma^r_{rr}}+\underbrace{\frac{1}{r^2}\left(-r-r^3\right)}_{b^{\theta\theta}\Gamma^r_{\theta\theta}}+\underbrace{\frac{1}{r}\left(-1-r^2\right)}_{b^{\theta\theta}\Gamma^r_{\phi\phi}}\right)\underbrace{\left(-\frac{2m}{r^5}\right)}_{\approx e_{rr}}\times\underbrace{4\pi r^2}_{\int dS_r},\\
&=\frac{-2\text{ m }r^2\left(r^2+1\right)\left(\dfrac{-r\left(r^2+1\right)}{r^2+1}+\dfrac{-r^3-r}{r^2}+\dfrac{-r^2-1}{r}\right)}{2r^5}.
\end{aligned}$$

If we cancel -2 and for all exponents, $\dfrac{a^n}{a^m}=a^{n-m}$ applied is

$$\frac{-2\text{ m }r^2\left(r^2+1\right)\left(\dfrac{-r\left(r^2+1\right)}{r^2+1}+\dfrac{-r^3-r}{r^2}+\dfrac{-r^2-1}{r}\right)}{2r^5} = -mr^{-3}\left(r^2+1\right)\left(-\frac{r\left(r^2+1\right)}{r^2+1}+\frac{-r^3-r}{r^2}+\frac{-r^2-1}{r}\right).$$

Cancel common terms in the numerator and denominator of $\frac{-r\left(r^2+1\right)}{r^2+1}$ where $\frac{-r\left(r^2+1\right)}{r^2+1} = \frac{r^2+1}{r^2+1} \times -r = -r$ is

$$-mr^{-3}\left(r^2+1\right)\left(-\frac{r\left(r^2+1\right)}{r^2+1}+\frac{-r^3-r}{r^2}+\frac{-r^2-1}{r}\right) = -\frac{m\left(r^2+1\right)\left(-r+\frac{-r^3-r}{r^2}+\frac{-r^2-1}{r}\right)}{r^3}.$$

Express $\frac{-r^3-r}{r^2}$ where $\frac{-r^3-r}{r^2} = -\frac{r^3}{r^2}-\frac{r}{r^2}$

$$\begin{aligned}
-\frac{m\left(r^2+1\right)\left(\frac{-r^2-1}{r}+\frac{-r^3-r}{r^2}-r\right)}{r^3} &= \frac{m\left(r^2+1\right)\left(-r+\left(-\frac{r^3}{r^2}-\frac{r}{r^2}\right)+\frac{-r^2-1}{r}\right)}{r^3}, \\
&= \frac{m\left(r^2+1\right)\left(-r-\frac{1}{r}+(-r)+\frac{-r^2-1}{r}\right)}{r^3}, \\
&= -\frac{m\left(r^2+1\right)}{r^3}\left(-r-\frac{1}{r}-r+\frac{-r^2-1}{r}\right), \\
&= -\frac{m\left(r^2+1\right)\left(-r-r-\frac{1}{r}+\left(-\frac{1}{r}-r\right)\right)}{r^3}, \\
&= -\frac{m\left(r^2+1\right)\left(-r-r-r-\frac{1}{r}-\frac{1}{r}\right)}{r^3}, \\
&= -\frac{m\left(r^2+1\right)}{r^3}\left((-r-r-r)+\left(-\frac{1}{r}-\frac{1}{r}\right)\right), \\
&= -\frac{m\left(-3r+\left(-\frac{1}{r}-\frac{1}{r}\right)\right)\left(r^2+1\right)}{r^3}, \\
&= -\frac{m\left(-3r-\frac{2}{r}\right)\left(r^2+1\right)}{r^3}, \\
&= -\frac{m}{r^3}\left(-3r^3-5r-\frac{2}{r}\right), \\
&= \frac{2m}{r^4}+\frac{5m}{r^2}+3m.
\end{aligned}$$

Putting back $\frac{1}{2}$

$$-\frac{m\left(r^2+1\right)\left(\frac{-r^2-1}{r}+\frac{-r^3-r}{r^2}-r\right)}{r^3} = \frac{2m}{2r^4}+\frac{5m}{2r^2}+\frac{3m}{2}$$

Now for Y we will see that it with its outer components equals $-\frac{m}{2}-\frac{m}{2r^2}$

$$-\frac{1}{16\pi}\int_{S_r}\left(1+r^2\right)b^{rr}\Gamma^r_{rr}e_{rr}dS_r = \frac{1}{2}\left(-\frac{1}{8\pi}\int_{S_r}\left(1+r^2\right)b^{rr}\Gamma^r_{rr}e_{rr}dS_r\right).$$

Momentarily we will leave out $\frac{1}{2}$

$$\left(-\frac{1}{8\pi}\int_{S_r}\left(1+r^2\right)b^{rr}\Gamma^r_{rr}e_{rr}dS_r\right)=\frac{1}{8\pi}\left(1+r^2\right)\underbrace{\left(-1-r^2\right)}_{b^{rr}\times -1}\left(-\frac{r}{1+r^2}\right)\times\underbrace{\frac{-2m}{r^5}}_{\approx e_{rr}}\times\underbrace{4\pi r^2}_{\int dS_r},$$
$$=-\frac{-\left(2r\right)\ \mathrm{m}\ r^2\left(r^2+1\right)\left(-r^2-1\right)}{2\left(r^2+1\right)r^5}.$$

Cancel common terms in the numerator and denominator of $\dfrac{-2m\left(-r\right)\left(r^2+1\right)\left(-r^2-1\right)r^2}{\left(r^2+1\right)r^5\times 2}$

$$\begin{aligned}
&\frac{1}{8\pi}\left(1+r^2\right)\left(-1-r^2\right)\left(-\frac{r}{1+r^2}\times\frac{-2m}{r^5}\times 4\pi r^2\right),\\
=&-\frac{-\left(2r\right)\ \mathrm{m}\ r^2\left(r^2+1\right)\left(-r^2-1\right)}{2\left(r^2+1\right)r^5},\\
=&-\frac{-\left(2r\right)\ \mathrm{m}\ r^2\left(r^2+1\right)\left(-r^2-1\right)}{2\left(r^2+1\right)r^5},\\
=&\frac{-2\left(-m\right)r\times r^2\left(-r^2-1\right)}{2r^5}.
\end{aligned}$$

Divide -2 in the numerator by 2 in the denominator

$$\begin{aligned}
-\frac{-\left(2r\right)\ \mathrm{m}\ r^2\left(r^2+1\right)\left(-r^2-1\right)}{2\left(r^2+1\right)r^5}=&\frac{mr\times r^2\left(-r^2-1\right)}{r^5},\\
=&mr^{-2}\left(-r^2-1\right),\\
=&\frac{-mr^2-m}{r^2}.
\end{aligned}$$

Putting back $\frac{1}{2}$

$$-\frac{1}{2}\times\frac{-\left(2r\right)\ \mathrm{m}\ r^2\left(r^2+1\right)\left(-r^2-1\right)}{2\left(r^2+1\right)r^5}=-\frac{m}{2}-\frac{m}{2r^2}$$

Now back to original equation, we finally see that $mass_{AH}$ equals m.

$$\begin{aligned}
&\lim_{r\to\infty}\frac{1}{16\pi}\int_{S_r}\left(\sqrt{1+r^2}\left(div^b e - \mathrm{d}\ tr^b e\right)+\left(tr^b e\right)d\sqrt{1+r^2}-e\left(\nabla^b\sqrt{1+r^2},\cdot\right)\right)(\nu_r)\,dS_r,\\
&=\lim_{r\to\infty}\int_{S_r}\Bigg(\underbrace{\sqrt{1+r^2}div^b e\,(\nu_r)}_{M}-\underbrace{\sqrt{1+r^2}\mathrm{d}\ tr^b e\,(\nu_r)}_{W}+\underbrace{tr^b e\cdot\nu_r\ d\sqrt{1+r^2}}_{X}\\
&\quad+\underbrace{\nu_r tr^b e\ d\sqrt{1+r^2}}_{Z}-\underbrace{e\left(\nabla^b\sqrt{1+r^2},\nu_r\right)}_{B}\Bigg)dS_r,\\
&=\lim_{r\to\infty}\left(\underbrace{\frac{2m}{2r^4}+\frac{5m}{2r^2}+\frac{3m}{2}-\frac{m}{2}-\frac{m}{2r^2}}_{M}-\underbrace{0}_{W}+\underbrace{0}_{X}+\underbrace{0}_{Z}+\underbrace{0}_{B}\right),\\
&=\left(\underbrace{\frac{3m}{2}-\frac{m}{2}}_{M}\right),\\
&=\frac{2m}{2},\\
&=m.
\end{aligned}$$

□

5.3. Graphs in Hyperbolic Space.

Theorem 5.1 (PMT for graphs over Hyperbolic space)**.** *Let $f:\mathbb{H}^n\to\mathbb{R}$ be an asymptotically hyperbolic function. Σ in $\mathbb{H}^{n+1}$ with induced metric g. Then*

$$mass_{AH}=\int_{\mathbb{H}^n\setminus\Omega}\frac{V\left[R_g+n(n-1)\right]}{\sqrt{1+V^2|df|^2}}dVol_g. \tag{5.7}$$

5.4. **Proof of PMT for Graphs over Hyperbolic space.** For the following set of calculations, I will first compute christoffel symbols of $\overline{b}$. Using this I compute second fundamental form of Σ in H^{n+1}. I compute the mean curvature of Σ in H^{n+1}, using Gauss (3.17) and find scalar curvature Σ in H^{n+1}, and use divergence theorem.

First, for computation for the Christoffel symbols of $\overline{b}$ on H^{n+1}, let's denote i,j as $a\in\left\langle 0,\underbrace{1,....,n}_{i,j}\right\rangle$ with the metric tensor

$$\begin{aligned}
\overline{b}&=b_{ij}dx^i\otimes dx^j+V^2ds\otimes ds+\overline{b}_{io}dx^i\times ds,\\
\overline{b}^{-1}&=b^{oj}\frac{\partial}{\partial x^j}\otimes\frac{\partial}{\partial x^i}+V^{-2}\frac{\partial}{\partial s}\otimes\frac{\partial}{\partial s}+\overline{b}^{oi}\frac{\partial}{\partial x^i}\otimes\frac{\partial}{\partial s}.
\end{aligned}\tag{5.8}$$

Visually the II fundamental form in this case is And we can see in II fundemental form that $\overline{\Gamma}^{o}_{oo}$

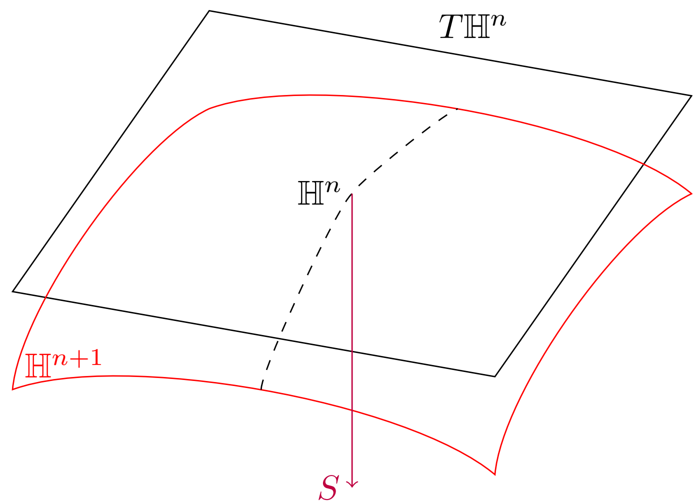


FIGURE 6. Visual of II fundamental form of Hyperbolic space

equals zero

$$\begin{aligned}
\overline{\Gamma}^o_{oo} &= \frac{1}{2}\overline{b}^{oa}\left(\partial_o\overline{b}_{ao} + \partial_o\overline{b}_{ao} - \partial_a\overline{b}_{oo}\right), \\
&= \frac{1}{2}\overline{b}^{oo}\left(\partial_o\overline{b}_{oo} + \partial_o\overline{b}_{oo} - \partial_o\overline{b}_{oo}\right) + \frac{1}{2}\overline{b}^{oi}\left(\partial_o\overline{b}_{io} + \partial_o\overline{b}_{io} - \partial_i\overline{b}_{oo}\right), \\
&= \frac{1}{2}\left(V^{-2}\right)\left(\partial_o\overline{b}_{oo}\right) + \frac{1}{2}(0)\left(\partial_o\overline{b}_{io} + \partial_o\overline{b}_{io} - \partial_i\overline{b}_{oo}\right), \\
&= \frac{1}{2}\left(V^{-2}\right)\left(\partial_o V^2\right), \\
&= \frac{1}{2}\left(V^{-2}\right)(0) \\
&= 0.
\end{aligned}$$

That $\overline{\Gamma}^i_{oo}$ equals $= -V\nabla^i V$ with $\partial_o V^2 = 0$ because V does not depend on S.

$$\begin{aligned}
\overline{\Gamma}^i_{oo} &= \frac{1}{2}\overline{b}^{ia}\left(\partial_o\overline{b}_{ao} + \partial_o\overline{b}_{ao} - \partial_a\overline{b}_{oo}\right), \\
&= \frac{1}{2}\overline{b}^{io}\left(\partial_o\overline{b}_{oo} + \partial_o\overline{b}_{oo} - \partial_o\overline{b}_{oo}\right) + \frac{1}{2}\overline{b}^{ii}\left(\partial_o\overline{b}_{io} + \partial_o\overline{b}_{io} - \partial_i\overline{b}_{oo}\right), \\
&= (0) + \frac{1}{2}(1)\left(-1\partial_i V^2\right), \\
&= \frac{1}{2}\left(-2V\partial_i V\right), \\
&= -V\nabla^i V.
\end{aligned}$$

That $\overline{\Gamma}^o_{io}$ equals $\dfrac{\nabla_i V}{V}$ with $\partial_i = \nabla^i$

$$\begin{aligned}
\overline{\Gamma}^o_{io} &= \frac{1}{2}\overline{b}^{oa}\left(\partial_o \overline{b}_{ai} + \partial_i \overline{b}_{ao} - \partial_a \overline{b}_{io}\right),\\
&= \frac{1}{2}\overline{b}^{oi}\left(\partial_o \overline{b}_{ii} + \partial_i \overline{b}_{io} - \partial_i \overline{b}_{io}\right) + \frac{1}{2}b^{oo}\left(\partial_o \overline{b}_{oi} + \partial_i \overline{b}_{oo} - \partial_o \overline{b}_{io}\right),\\
&= (0) + \frac{1}{2}\left(V^{-2}\right)\left(\partial_i V^2\right),\\
&= \frac{\nabla_i V^2}{V^2}\left(\frac{1}{2}\right),\\
&= \frac{\nabla_i V}{V}\left(\frac{1}{2}\right)\frac{2V}{V},\\
&= \frac{\nabla_i V}{V}.
\end{aligned}$$

That $\overline{\Gamma}^i_{jo}$ equals zero

$$\begin{aligned}
\overline{\Gamma}^i_{jo} &= \frac{1}{2}\overline{b}^{ia}\left(\partial_o \overline{b}_{aj} + \partial_i \overline{b}_{ao} - \partial_a \overline{b}_{jo}\right),\\
&= \frac{1}{2}\overline{b}^{io}\left(\partial_o \overline{b}_{oj} + \partial_i \overline{b}_{oo} - \partial_o \overline{b}_{jo}\right) + \frac{1}{2}\overline{b}^{ii}\left(\partial_o \overline{b}_{ij} + \partial_i \overline{b}_{io} - \partial_i \overline{b}_{jo}\right),\\
&= (0) + \frac{1}{2}(1)\left(\partial_o (0) + \partial_i (0) - \partial_i (0)\right),\\
&= 0.
\end{aligned}$$

That $\overline{\Gamma}^o_{ij}$ equals zero.

$$\begin{aligned}
\overline{\Gamma}^o_{ij} &= \frac{1}{2}\overline{b}^{oa}\left(\partial_j \overline{b}_{ai} + \partial_i \overline{b}_{aj} - \partial_a \overline{b}_{ij}\right),\\
&= \frac{1}{2}\overline{b}^{oo}\left(\partial_j \overline{b}_{oi} + \partial_i \overline{b}_{oj} - \partial_o \overline{b}_{ij}\right) + \frac{1}{2}\overline{b}^{oi}\left(\partial_j \overline{b}_{ii} + \partial_i \overline{b}_{ij} - \partial_i \overline{b}_{ij}\right),\\
&= \frac{1}{2}\left(V^{-2}\right)\left(\partial_j (0) + \partial_i (0) - \partial_o (0)\right) + \frac{1}{2}(0)\left(\partial_j \overline{b}_{ii} + \partial_i \overline{b}_{ij} - \partial_i \overline{b}_{ij}\right),\\
&= 0.
\end{aligned}$$

That $\overline{\Gamma}^k_{ij}$ equals $\frac{1}{2}\overline{b}^{ki}\left(\partial_j \overline{b}_{ii} + \partial_i \overline{b}_{ij} - \partial_i \overline{b}_{ij}\right) + \frac{1}{2}\overline{b}^{kj}\left(\partial_j \overline{b}_{ji} + \partial_i \overline{b}_{jj} - \partial_j \overline{b}_{ij}\right)$

$$\begin{aligned}
\overline{\Gamma}^k_{ij} &= \frac{1}{2}\overline{b}^{ka}\left(\partial_j \overline{b}_{ai} + \partial_i \overline{b}_{aj} - \partial_a \overline{b}_{ij}\right),\\
&= \frac{1}{2}\overline{b}^{ki}\left(\partial_j \overline{b}_{ii} + \partial_i \overline{b}_{ij} - \partial_i \overline{b}_{ij}\right) + \frac{1}{2}\overline{b}^{kj}\left(\partial_j \overline{b}_{ji} + \partial_i \overline{b}_{jj} - \partial_j \overline{b}_{ij}\right).
\end{aligned}$$

Referring back to (5.8) if we exclude s-coordinate, then we see that

$$\overline{b} = b_{ij}dx^i \otimes dx^j + \underbrace{V^2 ds \otimes ds}_{0},$$

$$\overline{b} = b_{ij}dx^i \otimes dx^j, \tag{5.9}$$

thus, $\overline{\Gamma}^k_{ij} = \Gamma^k_{ij}$ the induced metric on Σ is given by

$$g(X,Y) = \overline{b}\left(\overline{X},\overline{Y}\right) + V^2 \nabla_X f \nabla_Y f. \tag{5.10}$$

Lemma 5.2. *The second fundamental form S of Σ given by*

$$\begin{aligned} S\left(\overline{X},\overline{Y}\right) &= \frac{1}{|\overline{\nabla} F|}\overline{\nabla}^2_{\overline{x},\overline{y}}F, \\ &= \frac{V^2}{\sqrt{1+V^2|df|^2}}\left[\nabla^2_{X,Y}f + \frac{\nabla_X f\nabla_Y V + \nabla_X V\nabla_Y f}{V} + V\left\langle df, dV\right\rangle \nabla_X f\nabla_Y f\right]. \end{aligned} \tag{5.11}$$

Proof. Note that for graphs F definition of second fundamental form in (3.15) can be represented as a Hessian of F.

$$\overline{\nabla}^2_{\overline{X},\overline{Y}}F = \overline{\nabla}^2 F\left(X + \nabla_X f\partial_o, Y + \nabla_Y f\partial_o\right).$$

Distribute and associate $\overline{\nabla}^2$ with respected dimensions

$$\begin{aligned} \overline{\nabla}^2_{\overline{X},\overline{Y}}F &= \overline{\nabla}^2 F\left(X,Y\right) + \overline{\nabla}^2 F\left(X, \nabla_Y f\partial_o\right) + \overline{\nabla}^2\left(\nabla_X f\partial_o, Y\right) + \overline{\nabla}^2 F\left(\nabla_X f\partial_o, \nabla_Y f\partial_o\right), \\ &= \overline{\nabla}^2_{X,Y}F + \nabla_Y f\overline{\nabla}^2_{X,\partial_o}F + \nabla_X f\overline{\nabla}^2_{Y,\partial_o}F + \nabla_X f\nabla_Y f\overline{\nabla}^2_{X,\partial_o}F. \end{aligned}$$

We also see that

$$\overline{\nabla}^2_{\overline{X},\overline{Y}}F = \underbrace{X\left(\underbrace{\partial_Y F}_{-1}\right)}_{0} - \underbrace{\Gamma^k_{ij}}_{\Gamma^k_{ij}} X^i \underbrace{\partial_Y F}_{-1} - \underbrace{\overline{\Gamma^o_{ik}}}_{0} X^i\nabla jF = \Gamma^k_{ij},$$

$$\overline{\nabla}^2_{X,\partial_o}F = \underbrace{X\left(\underbrace{\partial_o F}_{-1}\right)}_{0} - \underbrace{\Gamma^o_{oi}}_{\frac{\nabla iV}{V}} X^i \underbrace{\partial_o F}_{-1} - \underbrace{\overline{\Gamma^j_{oi}}}_{0} X^i\nabla jF = \frac{\nabla_X V}{V},$$

$$\overline{\nabla}^2_{Y,\partial_o}F = \underbrace{Y\left(\underbrace{\partial_o F}_{-1}\right)}_{0} - \underbrace{\Gamma^o_{oi}}_{\frac{\nabla iV}{V}} Y^i \underbrace{\partial_o F}_{-1} - \underbrace{\overline{\Gamma^j_{oi}}}_{0} Y^i\nabla jF = \frac{\nabla_Y V}{V},$$

$$\overline{\nabla}^2_{X,\partial_o}F = \underbrace{X\left(\underbrace{\partial_o F}_{-1}\right)}_{0} - \underbrace{\Gamma^i_{oo}}_{V\nabla^i V} Y^i \underbrace{\partial_o F}_{-1} - \underbrace{\overline{\Gamma^o_{oo}}}_{0} X\nabla jF = V\nabla^i V.$$

Now going back to (5.11)

$$\begin{aligned} \overline{S}\left(\overline{X},\overline{Y}\right) = \overline{\nabla}^2_{\overline{X},\overline{Y}}F &= \frac{1}{|\overline{\nabla} F|}\left(\nabla_X\nabla_Y f + \frac{\nabla_X f\nabla_Y V}{V} + \frac{\nabla_X V\nabla_Y f}{V} + \nabla_X\nabla_Y V\left(df, dV\right)\right), \\ &= \frac{1}{\sqrt{V^{-2}+|df|^2}}\left[\nabla^2_{X,Y}f + \frac{\nabla_X f\nabla_Y V + \nabla_X V\nabla_Y f}{V} + V\left\langle df, dV\right\rangle \nabla_X f\nabla_Y f\right], \\ &= \frac{1}{V^{-2}\sqrt{1+V^2|df|^2}}\left[\nabla^2_{X,Y}f + \frac{\nabla_X f\nabla_Y V + \nabla_X V\nabla_Y f}{V} + V\left\langle df, dV\right\rangle \nabla_X f\nabla_Y f\right], \\ &= \frac{V^2}{\sqrt{1+V^2|df|^2}}\left[\nabla^2_{X,Y}f + \frac{\nabla_X f\nabla_Y V + \nabla_X V\nabla_Y f}{V} + V\left\langle df, dV\right\rangle \nabla_X f\nabla_Y f\right]. \end{aligned}$$

□

Next, we want to compute the mean curvature which is the trace of the second fundamental form. So we need the inverse metric g^{-1}.

$$df = \frac{\partial f}{\partial X_i} dx^i = \nabla_j f dx^i,$$
$$g = g_{ij} dx^i dx^j = \left(b_{ij} + V^2 \nabla_i f \nabla_j f\right) dx^i dx^j,$$

$$g^{-1} = b^{ij} - \frac{V^2 \nabla^i f \nabla^j f}{1 + V^2 |\nabla f|^2}, \quad \text{where } \nabla^i f = g^{ij} \nabla_j f = g^{ij} \frac{\partial f}{\partial X_j}. \tag{5.12}$$

We compute the mean curvature of Σ

Lemma 5.3. *The mean curvature of graph is*

$$\overline{H} = \frac{1}{|\overline{\nabla}|}\left[\Delta f - \frac{V^2 \langle \mathit{Hess}\ f, df \otimes df\rangle}{1 + V^2 |df|^2} + \left(1 + \frac{1}{1 + V^2|df|^2}\right)\left\langle df, \frac{dV}{V}\right\rangle\right]$$

Proof. Substituting from the definition of mean curvature (3.16) and using (5.11) and using g^1 from (5.12) as a reference for g^{ij} then

$$\begin{aligned}
\overline{H} =& g^{ij} S_{ij}, \\
=& \frac{1}{|\overline{\nabla}|}\left(b^{ij} - \frac{V^2 \nabla^i f \nabla^j f}{1 + V^2 |df|^2}\right) m \\
& \cdot \left[\nabla_i \nabla_j f + \frac{\nabla_i f \nabla_j V + \nabla_i V \nabla_j f}{V} + V \langle df, dV\rangle \nabla_i f \nabla_j f\right], \\
=& \frac{1}{|\overline{\nabla}|}\left[\Delta f + 2\left\langle df, \frac{dV}{V}\right\rangle + V\langle df, dV\rangle |df|^2\right., \\
& \left. - \frac{V^2}{1 + V^2|df|^2}\left(\langle \text{Hess f}, df \otimes df\rangle + 2|df|^2 \left\langle df, \frac{dV}{V}\right\rangle + V^2 |df|^4 \left\langle df, \frac{dV}{V}\right\rangle\right)\right], \\
=& \frac{1}{|\overline{\nabla}|}\left[\Delta f - \frac{V^2 \langle \text{Hess f}, df \otimes df\rangle}{1 + V^2|df|^2} + \frac{2 + V^2|df|2}{1 + V^2|df|^2}\left\langle df, \frac{dV}{V}\right\rangle\right], \\
=& \frac{1}{|\overline{\nabla}|}\left[\Delta f - \frac{V^2 \langle \text{Hess f}, df \otimes df\rangle}{1 + V^2|df|^2} + \left(\frac{1 + V^2|df|^2}{1 + V^2|df|^2} + \frac{1}{1 + V^2|df|^2}\right)\left\langle df, \frac{dV}{V}\right\rangle\right], \\
=& \frac{1}{|\overline{\nabla}|}\left[\Delta f - \frac{V^2 \langle \text{Hess f}, df \otimes df\rangle}{1 + V^2|df|^2} + \left(1 + \frac{1}{1 + V^2|df|^2}\right)\left\langle df, \frac{dV}{V}\right\rangle\right].
\end{aligned}$$

□

5.5. **Norm of II fundamental form of Σ.**

Lemma 5.4. *The norm of the II fundamental form of Σ is given by*

$$\left.\begin{aligned}
|\overline{S}_g^2| =& \frac{-V^2}{1 + V^2|df|^2}\left[|\mathit{Hess}\ f|^2 + 2|df|^2 |\frac{dV}{|V|^2} + 2\left\langle df, \frac{dV}{V}\right\rangle^2 + V^4 |df|^4 \left\langle df, \frac{dV}{V}\right\rangle^2\right. \\
& \left. + 4\langle \mathit{Hess}\ f, df \otimes df\rangle + 2V^2 \left\langle df, \frac{dV}{V}\right\rangle \langle \mathit{Hess}\ f, df \otimes df\rangle + 4V^2 |df|^2 \left\langle df, \frac{dV}{V}\right\rangle^2\right],
\end{aligned}\right\} A$$

$$\left.\begin{aligned}&\frac{+2V^4}{(1+V^2|df|^2)^2}\Bigg[|Hess\, f(\nabla f,\cdot)\,|^2+\left(1+V^2|df|^2\right)^2\left\langle df,\frac{dV}{V}\right\rangle^2|df|^2\\&+|df|^4|\frac{dV}{V}|^2+2\left(1+V^2|df|^2\right)Hess\, f(\nabla f,\nabla f)\left\langle df,\frac{dV}{V}\right\rangle\\&+2|df|^2\left\langle Hess\, f,\nabla f\otimes\frac{\nabla V}{V}\right\rangle+2\left(1+V^2|df|^2\right)|df|^2\left\langle df,\frac{dV}{V}\right\rangle^2\Bigg],\end{aligned}\right\}B$$

$$\left.-\frac{V^2}{1+V^2|df|^2}\left[\frac{V^2\langle Hess\, f,df\otimes df\rangle}{1+V^2|df|^2}+\left(1+\frac{1}{1+V^2|df|^2}V^2\right)|df|^2\left\langle df,\frac{dV}{V}\right\rangle\right]^2.\ \right\}(C)$$

Proof. To begin with, we know the normalization of II fundamental form is of the product of the inverse of the induced metric times the the second fundamental form, as follows

$$|\overline{S}_g^2|=g^{ik}g^{jl}\overline{S}_{ij}\overline{S}_{kl},$$

where (5.12) is substituted in and the following components are broken into three parts

$$\begin{aligned}|\overline{S}_g^2|&=\left(b^{ik}-\frac{V^2\nabla^i f\nabla^k f}{1+V^2|df|^2}\right)\left(b^{jl}-\frac{V^2\nabla^j f\nabla^l f}{1+V^2|df|^2}\right)\overline{S}_{ij}\overline{S}_{kl},\\&=b^{ik}b^{jl}\overline{S}_{ij}\overline{S}_{kl}-2\frac{V^2b^{ik}\nabla^j f\nabla^l f}{1+V^2|df|^2}\overline{S}_{ij}\overline{S}_{kl}|\frac{V^4\nabla^i f\nabla^j f\nabla^k f\nabla^l f}{(1+V^2|df|^2)^2}\overline{S}_{ij}\overline{S}_{kl},\\&=\underbrace{|\overline{S}|_b^2}_{A}-\underbrace{2\frac{V^2b^{ik}\nabla^j f\nabla^l f}{1+V^2|df|^2}\overline{S}_{ij}\overline{S}_{kl}}_{B}+\underbrace{\left(\frac{V^2\overline{S}\left(\nabla f,\nabla f\right)}{1+V^2|df|^2}\right)^2}_{(C)}.\end{aligned}\tag{5.13}$$

The calculation is rather complicated so it will be broken into three parts:A,B,C

$$A=b^{ik}b^{jl}\overline{S}_{ij}\overline{S}_{kl},$$

using what we know about $\overline{S}_{ij}$

$$\begin{aligned}A=&b^{ik}b^{jl}\left(\frac{V}{\sqrt{1+V^2|df|^2}}\left[\nabla^2_{i,j}f+\frac{\nabla_i f\nabla_j V+\nabla_i V\nabla_j f}{V}+V\left\langle df,dV\right\rangle\nabla_i f\nabla_j f\right]\right),\\&\cdot\left(\frac{V}{\sqrt{1+V^2|df|^2}}\left[\nabla^2_{k,l}+\frac{\nabla_k f\nabla_l V+\nabla_k V\nabla_l f}{V}+V\left\langle df,dV\right\rangle\nabla_k f\nabla_l f\right]\right),\\=&b^{ik}b^{jl}\frac{V^2}{1+V^2|df|^2}\Bigg[\nabla^2_{i,j}f\nabla^2_{k,l}f+\frac{\nabla^2_{i,j}f\left(\nabla_k f\nabla_l V+\nabla_k V\nabla_l f\right)}{V},\\&+\nabla^2_{i,j}fV\left\langle df,dV\right\rangle\nabla_k f\nabla_l f+\nabla^2_{k,l}f\left(\nabla_i f\nabla_j V+\nabla_i V\nabla_k f\right)\frac{1}{V},\\&+\left(\nabla_i f\nabla_j V\nabla_k f\nabla_l V+\nabla_i f\nabla_j V\nabla_k V\nabla_l f\right),\\&+\nabla_i V\nabla_j f\nabla_k f\nabla_l V+\nabla_i V\nabla_j f\nabla_k V\nabla_l f)\frac{1}{V^2},\\&\nabla^2_{k,l}V\left\langle df,dV\right\rangle\nabla_l f\nabla_j f+V\left\langle df,dV\right\rangle\nabla_i f\nabla_j f\left(\frac{\nabla_k f\nabla_l V+\nabla_k V\nabla_l f}{V}\right),\\&+\left(V\left\langle df,dV\right\rangle\nabla_i f\nabla_j f\right)\left(V\left\langle df,dV\right\rangle\nabla_k f\nabla_l f\right)\Big],\end{aligned}$$

distributing $b^{ik}b^{jl}$ with its respective Laplace

$$\begin{aligned}A =&\frac{V^2}{1+V^2|df|^2}\Bigg[\nabla_k\nabla_l f\nabla_k\nabla_l f + \frac{\nabla_k\nabla_l f\left(2\nabla_l f\nabla_k V\right)}{V} + \nabla_k\nabla_l f V\left\langle df, dV\right\rangle\nabla_k f\nabla_l f\\ &+\frac{\nabla_k\nabla_l f\left(2\nabla_k f\nabla_l V\right)}{V} + \left(4\nabla_k f\nabla_l f\nabla_k V\nabla_l V\right)\frac{1}{V^2} + \nabla_k\nabla_l f V\left\langle df, dV\right\rangle\nabla_k f\nabla_l f\\ &+ V\left\langle df dV\right\rangle\nabla_k\nabla_l f\frac{\nabla_k f\left(2\nabla_k f\nabla_l V\right)}{V}\\ &+\left(\nabla_k f\nabla_l V + \nabla_k V\nabla_l f\right)\frac{1}{V}\cdot V\left\langle df, dV\right\rangle\nabla_k f\nabla_l f + \left(V\left\langle df, dV\right\rangle\nabla_k f\nabla_l f\right)^2\Bigg],\end{aligned}$$

rearranged we get 9 components separated by plus signs

$$\begin{aligned}A =&\frac{V^2}{1+V^2|df|^2}\Bigg[\underbrace{|\text{Hess f}|^2}_{I_2} + \underbrace{\frac{\text{Hess f}\left(2df dV\right)}{V}}_{I_6} + \underbrace{\text{Hess f } V\left\langle df, dV\right\rangle|df|^2}_{I_3}\\ &+\underbrace{\frac{\text{Hess f }\left(2df dV\right)}{V}}_{I_5} + \underbrace{\left(4|df|^2 dV^2\right)\frac{1}{V^2}}_{I_1} + \underbrace{\text{Hess f } V\left\langle df, dV\right\rangle|df|^2}_{I_4} + \underbrace{V\left\langle df, dV\right\rangle|df|^2\left(2\left\langle df, \frac{dV}{V}\right\rangle\right)}_{I_8}\\ &+\underbrace{\left(df dV + df dV\right)\frac{1}{V}\cdot V\left\langle df, dV\right\rangle|df|^2}_{I_7} + \underbrace{\left(V\left\langle df, dV\right\rangle|df|^2\right)^2}_{I_9}\Bigg].\end{aligned}$$

Observe that

$$\begin{aligned}&V^2\left\langle df, \frac{dV}{V}\right\rangle = V\left\langle df, dV\right\rangle,\\ &|df|^2 := df\otimes df.\end{aligned}$$

Now we expand I_1

$$I_1 = \underbrace{\frac{1}{V^2}\left(4|df|^2 dV^2\right)}_{I_1} := 2|df|^2\left(\frac{dV}{V}\right)^2 + 2\left\langle df, \frac{dV}{V}\right\rangle^2.$$

Next, $I_3 + I_4$ is as follows

$$\begin{aligned}I_3 + I_4 :&\underbrace{\text{Hess f V}\left\langle df, dV\right\rangle|df|^2 + \text{ Hess f V}\left\langle df, dV\right\rangle|df|^2}_{I_3 \text{ and } I_4},\\ &= \text{Hess f } V^2\left\langle df, \frac{dV}{V}\right\rangle|df|^2 + \text{ Hess f } V^2\left\langle df, \frac{dV}{V}\right\rangle|df|^2,\\ &= 2V^2\left\langle df, \frac{dV}{V}\right\rangle\left\langle\text{Hess f}, df\otimes df\right\rangle.\end{aligned}$$

Similarily $I_5 + I_6$ is

$$\begin{aligned}I_5 + I_6 &= \underbrace{\text{Hess f}\left(2df\frac{dV}{V}\right) + \text{Hess f}\left(2df\frac{dV}{V}\right)}_{I_5 \text{ and } I_6},\\ &= 2\left\langle\text{Hess f}, df\otimes\frac{dV}{V}\right\rangle + 2\left\langle\text{Hess f}, df\otimes\frac{dV}{V}\right\rangle,\\ &= 4\left\langle\text{Hess f}, df\otimes\frac{dV}{V}\right\rangle.\end{aligned}$$

Finally, $I_7+I_8+I_9$ can be rewritten as

$$I_7+I_8+I_9 = \underbrace{(df dV + df dV)\frac{1}{V}\cdot V\left\langle df, dV\right\rangle |df|^2 + V\left\langle df, dV\right\rangle |df|^2 (2\left\langle df, \frac{dV}{V}\right\rangle)}_{I_7+I_8}$$

$$+ \underbrace{\left(V\left\langle df, dV\right\rangle |df|^2\right)^2}_{I_9},$$

$$= 2V^2\left\langle df, \frac{dV}{V}\right\rangle^2 |df|^2 + 2V\left\langle df, dV\right\rangle |df|^2\left\langle df, \frac{dV}{V}\right\rangle + \left(V\left\langle df, dV\right\rangle |df|^2\right)^2,$$

$$= 4V^2|df|^2\left\langle df, dV\right\rangle^2 + V^4|df|^4\left\langle df, \frac{dV}{V}\right\rangle^2$$

Combining $I_1+...+I_9$ we have

$$A = \frac{V^2}{1+V^2|df|^2}\left[\underbrace{|\text{Hess f}|^2}_{I_2} + \underbrace{2|df|^2|\frac{dV}{V}|^2| + 2\left\langle df, \frac{dV}{V}\right\rangle^2}_{I_1} + \underbrace{V^4|df|^4\left\langle df, \frac{dV}{V}\right\rangle^2}_{I_9}\right.$$

$$\left. + \underbrace{4\left\langle \text{Hess f}, df\otimes\frac{dV}{V}\right\rangle}_{I_5+I_6} + \underbrace{2V^2\left\langle df, \frac{dV}{V}\right\rangle\left\langle \text{Hess f}, df\otimes df\right\rangle}_{I_3+I_4} + \underbrace{4V^2|df|^2\left\langle df, \frac{dV}{V}\right\rangle^2}_{I_7+I_8}\right].$$

Now for the B component

$$B = \frac{-2V^2b^{ik}}{1+V^2|df|^2}\nabla^j f\overline{S}_{ij}\nabla^l f\overline{S}_{kl},$$

$$= \frac{-2V^2b^{ik}}{1+V^2|df|^2}\nabla^j f\nabla^l f\overline{S}_{ij}\overline{S}_{kl}.$$

Using Lemma (5.2), the (5.2) implies that substituting $\overline{S}_{ij}\overline{S}_{kl}$ for (5.11)

$$B = \frac{-2V^2b^{ik}}{1+V^2|df|^2}\nabla^j f\nabla^l f\frac{1}{|\overline{\nabla}F|}\left[\nabla_i\nabla_j f + \frac{\nabla_i f\nabla_j fV + \nabla_i V\nabla_j f}{V} + V\left\langle df, dV\right\rangle\nabla_l f\nabla_j f\right]$$

$$\cdot\frac{1}{|\overline{\nabla}F|}\left[\nabla_k\nabla_l f + \frac{\nabla_k f\nabla_l V + \nabla_k V\nabla_l f}{V} + V\left\langle df, dV\right\rangle\nabla_k f\nabla_l f\right].$$

Expanding out B

$$B = \frac{-2V^2b^{ik}}{1+V^2|df|^2}\nabla^j f\nabla^l f\frac{1}{|\overline{\nabla}F|^2}\left[\nabla_i\nabla_j f\nabla_k\nabla_l f + \nabla_i\nabla_j f\frac{(\nabla_k f\nabla_l V + \nabla_k V\nabla_l f)}{V}\right.$$

$$+\nabla_i\nabla_j fV\left\langle df, dV\right\rangle\nabla_k f\nabla_l f$$

$$+\left(\nabla_i f\nabla_j V\nabla_k f\nabla_l V + \nabla_i f\nabla_j V\nabla_k V\nabla_l f + \nabla_i V\nabla_j f\nabla_k f\nabla_l V + \nabla_i V\nabla_j f\nabla_k V\nabla_l f\right)\frac{1}{V^2}$$

$$+\left(\nabla_i f\nabla_j V\nabla_k\nabla_l f\right)\frac{1}{V} + \left(\nabla_i\nabla_j f\nabla_k\nabla_l f\right)\frac{1}{V}$$

$$+\left\langle df, dV\right\rangle\nabla_k f\nabla_l f\left(\nabla_i f\nabla_j V + \nabla_i V\nabla_j f\right) + V\left\langle df, dV\right\rangle\nabla_i f\nabla_j f\nabla_k\nabla_l f$$

$$\left. +V\left\langle df, dV\right\rangle\nabla_i f\nabla_j f\frac{(\nabla_k f\nabla_l V + \nabla_k V\nabla_l f)}{V} + V\left\langle df, dV\right\rangle V\left\langle df, dV\right\rangle\nabla_i f\nabla_j f\nabla_k f\nabla_l f\right].$$

Distributing $\nabla^j f \nabla^l f \dfrac{1}{|\overline{\nabla} F|^2}$ and labeling components by the letter J

$$
\begin{aligned}
B = \frac{-2V^2}{1+V^2|df|^2}\frac{1}{|\overline{\nabla} F|^2}\Bigg[& \underbrace{\nabla_k f \nabla_l f \nabla_k \nabla_j f \nabla_j \nabla_l f}_{J_1} + \underbrace{\nabla_k f \nabla_l f \nabla_k \nabla_j f \frac{(\nabla_j f \nabla_l V + \nabla_j V \nabla_l f)}{V}}_{J_2} \\
& \underbrace{+\nabla_k f \nabla_l f \nabla_k \nabla_j f V \left\langle df, dV \right\rangle \nabla_j f \nabla_l f}_{J_3} \\
& + \underbrace{\nabla_k f \nabla_l f \left(\nabla_k f \nabla_j V \nabla_j f \nabla_l V + \nabla_k f \nabla_j V \nabla_j V \nabla_l f + \nabla_k V \nabla_j f \nabla_j f \nabla_l V + \nabla_k V \nabla_j f \nabla_j V \nabla_l f\right)\frac{1}{V^2}}_{J_4} \\
& + \underbrace{\nabla_k f \nabla_l f \left(\nabla_j f \nabla_l V \nabla_j \nabla_k f\right)\frac{1}{V}}_{J_3} + \underbrace{\nabla_k f \nabla_l f \left(\nabla_j \nabla_l f \nabla_j V \nabla_k f\right)\frac{1}{V}}_{J_3} \\
& + \underbrace{\nabla_k f \nabla_l f \left\langle df, dV \right\rangle \nabla_j f \nabla_k f \left(\nabla_j f \nabla_l V + \nabla_j V \nabla_l f\right)}_{J_4} + \underbrace{\nabla_k f \nabla_l f V \left\langle df, dV \right\rangle \nabla_j f \nabla_l f \nabla_k \nabla_j f}_{J_3} \\
& \underbrace{+\nabla_j f \nabla_l f V \left\langle df, dV \right\rangle \nabla_j f \nabla_k f \frac{(\nabla_k f \nabla_l V + \nabla_k V \nabla_l f)}{V} + \nabla_k f \nabla_l f V \left\langle df, dV \right\rangle V \left\langle df, dV \right\rangle \nabla_j f \nabla_l f \nabla_j f \nabla_k f \Bigg]}_{J_4}.
\end{aligned}
$$

Now for calculating each expanded component, first we expand J_1 by the definition of hessian from from (4.30)

$$
\begin{aligned}
J_1 &= \nabla_k f \nabla_l f \nabla_k \nabla_j f \nabla_j \nabla_l f, \\
&= \nabla_k \nabla_l f \nabla_k f \nabla_j \nabla_j f \nabla_l f, \\
&= \text{Hess f}\,(f,\cdot)\,\text{Hess f}\,(f,\cdot)\,, \\
&= |\text{Hess f}\,(f,\cdot)\,|^2.
\end{aligned}
$$

Now for component J_2, we will distribute and again use (4.30) as a substitute

$$
\begin{aligned}
J_2 = {} & \nabla_k f \nabla_l f \nabla_k \nabla_j f \frac{(\nabla_j f \nabla_l V + \nabla_j V \nabla_l f)}{V}\ , \\
= {} & \underbrace{\left(\nabla_k f \nabla_l f \nabla_k \nabla_j f \nabla_j f \nabla_l V\right)\frac{1}{V} + \left(\nabla_k f \nabla_l f \nabla_k \nabla_j f \nabla_j f \nabla_l V\right)\frac{1}{V}}_{\text{Distributing } \frac{1}{V}}, \\
= {} & \left(|\nabla_k f|^2 \nabla_l \nabla_j f \nabla_j f \nabla_l V\right)\frac{1}{V} + \left(|\nabla_k f|^2 \nabla_l \nabla_j f \nabla_j f \nabla_l V\right)\frac{1}{V}, \\
= {} & \underbrace{\left(|\nabla f|^2 \left\langle \text{Hess f}, df \otimes \nabla V \right\rangle\right)\frac{1}{V} + \left(|\nabla f|^2 \left\langle \text{Hess f}, df \otimes \nabla V \right\rangle\right)\frac{1}{V}}_{\text{Using the definition of Hess}}, \\
= {} & 2\left(|\nabla f|^2 \left\langle \text{Hess f}, df \otimes \frac{\nabla V}{V} \right\rangle\right).
\end{aligned}
$$

Now for J_3 components we will again use (4.30),and then redistribute

$$
\begin{aligned}
J_3 + J_3 + J_3 + J_3 &= \nabla_k f \nabla_l f \nabla_k \nabla_j f V \langle df, dV \rangle \nabla_j f \nabla_l f \\
&+ \nabla_k f \nabla_l f \left(\nabla_j f \nabla_l V \nabla_j \nabla_k f \right) \frac{1}{V} \\
&+ \nabla_k f \nabla_l f \left(\nabla_j \nabla_l f \nabla_j V \nabla_k f \right) \frac{1}{V} \\
&+ \nabla_k f \nabla_l f V \langle df, dV \rangle \nabla_j f \nabla_l f \nabla_k \nabla_j f, \\
&= \underbrace{|df|^2 V \langle df, dV \rangle \, \text{Hess f} \left(\nabla f, \nabla f \right)}_{\text{rewritten in coordinate-free notation}} \\
&+ \text{Hess f} \left(\nabla f, \nabla f \right) \left(2 \left\langle df, \frac{dV}{V} \right\rangle \right) \\
&+ \text{Hess f} \left(\nabla f, \nabla f \right) V \left\langle df, dV \right) |df|^2, \\
&= \text{Hess f} \left(\nabla f, \nabla f \right) V^2 \left\langle df, \frac{dV}{V} \right\rangle |df|^2 \\
&+ 2\text{Hess f} \left(\nabla f, \nabla f \right) \left\langle df, \frac{dV}{V} \right\rangle \\
&+ \text{Hess f} \left(\nabla f, \nabla f \right) V^2 \langle df, dV \rangle |df|^2, \\
&= 2\text{Hess f} \left(\nabla f, \nabla f \right) V^2 \left\langle df, \frac{dV}{V} \right\rangle |df|^2 \\
&+ 2\text{Hess f} \left(\nabla f, \nabla f \right) \left\langle df, \frac{dV}{V} \right\rangle, \\
&= 2 \left(1 + V^2 |df|^2 \right) \text{Hess f} \left(\nabla f, \nabla f \right) \left\langle df, \frac{dV}{V} \right) .
\end{aligned}
$$

Now for J_4 components, we add, distribute and fit some elements into a quadratic form to match the final answer later

$$
\begin{aligned}
J_4 + J_4 + J_4 =& \nabla_k f \nabla_l f \left(\nabla_k f \nabla_j V \nabla_j f \nabla_l V + \nabla_k f \nabla_j V \nabla_j V \nabla_l f + \nabla_k V \nabla_j f \nabla_j f \nabla_l V + \nabla_k V \nabla_j f \nabla_j V \nabla_l f\right) \frac{1}{V^2} \\
&+ \nabla_k f \nabla_l f \left\langle df, dV \right\rangle \nabla_j f \nabla_k f \left(\nabla_j f \nabla_l V + \nabla_j V \nabla_l f\right) \\
&+ \nabla_j f \nabla_l f V \left\langle df, dV \right\rangle \nabla_j f \nabla_k f \frac{\left(\nabla_k f \nabla_l V + \nabla_k V \nabla_l f\right)}{V} \\
&+ \nabla_k f \nabla_l f V \left\langle df, dV \right\rangle V \left\langle df, dV \right\rangle \nabla_j f \nabla_l f \nabla_j f \nabla_k f, \\
=& \nabla_k f \nabla_l f \left(4 \nabla_k f \nabla_j V \nabla_j f \nabla_l V\right) \frac{1}{V^2} \\
&+ \nabla_k f \nabla_l f \left(2 \nabla_j f \nabla_k V \left\langle df, dV \right\rangle \nabla_j f \nabla_l f\right) \\
&+ \nabla_j f \nabla_l f \left(\left(2 \nabla_k f \nabla_l V\right) V \left\langle df, dV \right\rangle \nabla_j f \nabla_k f\right) \frac{1}{V} \\
&+ \nabla_k f \nabla_l f V^2 \left\langle df, dV \right\rangle^2 \nabla_j f \nabla_l f \nabla_j f \nabla_k f, \\
=& 3|df|^2 \left\langle df, \frac{dV}{V} \right\rangle^2 + |df|^2 \left\langle df, \frac{dV}{V} \right\rangle^2 \textit{ split 4} \to \textit{3+1} \\
&+ 2|df|^2 V^2 \left\langle df, \frac{dV}{V} \right\rangle^2 |df|^2 \\
&+ 2|df|^2 V^2 \left\langle df, \frac{dV}{V} \right\rangle^2 |df|^2 \\
&+ |df|^6 V^4 \left\langle df, \frac{dV}{V} \right\rangle^2, \\
=& 3|df|^2 \left\langle df, \frac{dV}{V} \right\rangle^2 + |df|^2 \left\langle df, \frac{dV}{V} \right\rangle^2 + 2|df|^2 V^2 \left\langle df, \frac{dV}{V} \right\rangle^2 |df|^2 + |df|^6 V^4 \left\langle df, \frac{dV}{V} \right\rangle^2 \\
&+ 2|df|^2 V^2 \left\langle df, \frac{dV}{V} \right\rangle^2 |df|^2, \\
=& 3|df|^2 \left\langle df, \frac{dV}{V} \right\rangle^2 + \left(1 + 2V^2|df|^2 + V^4|df|^4\right) |df|^2 \left\langle df, \frac{dV}{V} \right\rangle^2 \\
&+ 2|df|^2 V^2 \left\langle df, \frac{dV}{V} \right\rangle^2 |df|^2, \\
=& 3|df|^2 \left\langle df, \frac{dV}{V} \right\rangle^2 + \underbrace{\left(1 + V^2|df|\right)^2 |df|^2 \left\langle df, \frac{dV}{V} \right\rangle^2}_{\text{squared form}} \\
&+ 2|df|^2 V^2 \left\langle df, \frac{dV}{V} \right\rangle^2 |df|^2, \\
=& \underbrace{2|df|^2 \left\langle df, \frac{dV}{V} \right\rangle^2 + |df|^2 \left\langle df, \frac{dV}{V} \right\rangle^2}_{\textit{split } 3 \to 2+1} + \left(1 + V^2|df|\right)^2 |df|^2 \left\langle df, \frac{dV}{V} \right\rangle^2 \\
&+ 2|df|^2 V^2 \left\langle df, \frac{dV}{V} \right\rangle^2 |df|^2, \\
=& \underbrace{2|df|^2 \left\langle df, \frac{dV}{V} \right\rangle^2 + 2|df|^2 V^2 \left\langle df, \frac{dV}{V} \right\rangle^2 |df|^2}_{\textit{reordered}}
\end{aligned}
$$

$$
\begin{aligned}
&+|df|^2\left\langle df\frac{dV}{V}\right\rangle^2+\left(1+V^2|df|\right)^2|df|^2\left\langle df,\frac{dV}{V}\right\rangle^2,\\
&=\underbrace{2\left(1+V^2|df|^2\right)|df|^2\left\langle df,\frac{dV}{V}\right\rangle^2}_{\textit{Factored out }|df|^2\left\langle df,\frac{dV}{V}\right\rangle^2}\\
&+|df|^2\left(\nabla f,\frac{\nabla V}{V}\right)^2+\left(1+V^2|df|\right)^2|df|^2\left\langle df,\frac{dV}{V}\right\rangle^2,\\
&=2\left(1+V^2|df|^2\right)|df|^2\left\langle df,\frac{dV}{V}\right\rangle^2\\
&+|df|^2\left(\Delta f\Delta f\frac{\nabla V\nabla V}{VV}\right)+\left(1+V^2|df|\right)^2|df|^2\left\langle df,\frac{dV}{V}\right\rangle^2,\\
&=2\left(1+V^2|df|^2\right)|df|^2\left\langle df,\frac{dV}{V}\right\rangle^2\\
&+\underbrace{|df|^2|df|^2|\frac{dV}{V}|^2+\left(1+V^2|df|\right)^2|df|^2\left\langle df,\frac{dV}{V}\right\rangle^2}_{\nabla f\textit{ rewritten as }df},\\
&=2\left(1+V^2|df|^2\right)|df|^2\left\langle df,\frac{dV}{V}\right\rangle^2\\
&+|df|^4|\frac{dV}{V}|^2+\left(1+V^2|df|\right)^2|df|^2\left\langle df,\frac{dV}{V}\right\rangle^2.
\end{aligned}
$$

All together we get our final answer for B

$$
\begin{aligned}
B=\frac{-2V^2}{1+V^2|df|^2}\frac{1}{|\overline{\nabla}F|^2}\Bigg[&\underbrace{|\text{Hess f}\,(f,\cdot)\,|^2}_{J_1}+\underbrace{2\left(1+V^2|df|^2\right)|df|^2\left\langle df,\frac{dV}{V}\right\rangle^2}_{J_4}\\
&+\underbrace{|df|^4|\frac{dV}{V}|^2+\left(1+V^2|df|\right)^2|df|^2\left\langle df,\frac{dV}{V}\right\rangle^2}_{J_4+J_4}+\underbrace{2\left(1+V^2|df|^2\right)\text{Hess f}\,(\nabla f,\nabla f)\left\langle df,\frac{dV}{V}\right)}_{J_3}\\
&+\underbrace{2\left(|\nabla f|^2\left\langle \text{Hess f}, df\otimes\frac{\nabla V}{V}\right\rangle\right)}_{J_2}\Bigg].
\end{aligned}
$$

Now for the C component

$$
C=\left(\frac{V^2\overline{S}\,(\nabla_i f,\nabla_j f)}{1+V^2|df|^2}\right)^2,
$$

using (5.11)

$$
\begin{aligned}
C =&\frac{V^4}{(1+V^2|df|^2)^2}\left[\frac{V}{\sqrt{1+V^2|df|^2}}\nabla_i f\nabla_j f\nabla_i\nabla_j f\right. \\
&\left.+\frac{\nabla_i f\nabla_j f\left(2\nabla_i f\nabla_j V\right)}{V}+\nabla_i f\nabla_j fV\left\langle df, dV\right\rangle\nabla_i f\nabla_j f\right]^2.
\end{aligned} \tag{5.14}
$$

Distributing out the square, we will break down the calculations for (C) as $P_1+P_2+P_3+P_4+P_5+P_6+P_7+P_8+P_9$

$$
\begin{aligned}
=&\frac{V^6}{(1+V^2|df|^2)^3}[\underbrace{\nabla_i f\nabla_j f\nabla_i\nabla_j f\nabla_i f\nabla_j f\nabla_i\nabla_j f}_{P_1}+\underbrace{\nabla_i f\nabla_j f\nabla_i\nabla_j f\nabla_i f\nabla_j f\left(2\nabla_i f\nabla_j V\right)\frac{1}{V}}_{P_2} \\
&+\underbrace{\nabla_i f\nabla_j f\nabla_i\nabla_j f\nabla_i f\nabla_j fV\left\langle df, dV\right\rangle\nabla_i f\nabla_j f}_{P_3} \\
&+\underbrace{\nabla_i f\nabla_j f\left(2\nabla_i f\nabla_j V\right)\frac{1}{V}\nabla_i f\nabla_j f\nabla_i\nabla_j f}_{P_7}+\underbrace{\nabla_i f\nabla_j f\left(2\nabla_i f\nabla_j V\right)\frac{1}{V}\nabla_i f\nabla_j f\left(2\nabla_i f\nabla_j V\right)\frac{1}{V}}_{P_4} \\
&+\underbrace{\nabla_i f\nabla_j f\left(2\nabla_i f\nabla_j V\right)\frac{1}{V}\nabla_i f\nabla_j fV\left\langle df, dV\right\rangle\nabla_i f\nabla_j f}_{P_9} \\
&+\underbrace{\nabla_i f\nabla_j fV\left\langle df, dV\right\rangle\nabla_i f\nabla_j f\nabla_i f\nabla_j f\nabla_i\nabla_j f}_{P_8} \\
&+\underbrace{\nabla_i f\nabla_j fV\left\langle df, dV\right\rangle\nabla_i f\nabla_j f\nabla_i f\nabla_j f\left(2\nabla_i f\nabla_j V\right)\frac{1}{V}}_{P_5} \\
&+\underbrace{\nabla_i f\nabla_j fV\left\langle df, dV\right\rangle\nabla_i f\nabla_j f\nabla_i f\nabla_j fV\left\langle df, dV\right\rangle\nabla_i f\nabla_j f]}_{P_6}.
\end{aligned}
$$

First for P_1 we substitute using (4.30)

$$
\begin{aligned}
P_1 &= \nabla_i f\nabla_j f\nabla_i\nabla_j f\nabla_i f\nabla_j f\nabla_i\nabla_j f, \\
&= \left\langle \text{Hess f}, df\otimes df\right\rangle\left\langle \text{Hess f}, df\otimes df\right\rangle, \\
&= |\left\langle \text{Hess f}, df\otimes df\right\rangle|^2.
\end{aligned}
$$

For P_2 we substitute using (4.29)

$$
\begin{aligned}
P_2 &= \nabla_i f\nabla_j f\nabla_i\nabla_j f\nabla_i f\nabla_j f\left(2\nabla_i f\nabla_j V\right)\frac{1}{V}, \\
&= 2|df|^2\left\langle \text{Hess f}, df\otimes df\right\rangle\left\langle df, \frac{dV}{V}\right\rangle.
\end{aligned}
$$

For P_3 we distribute around and then substitute using (4.29)

$$
\begin{aligned}
P_3 &= \nabla_i f\nabla_j f\nabla_i\nabla_j f\nabla_i f\nabla_j fV\left\langle df, dV\right\rangle\nabla_i f\nabla_j f, \\
&= \nabla_i f\nabla_j f\nabla_i\nabla_j fV^2|df|^2|df|^2\left\langle df, \frac{dV}{V}\right\rangle, \\
&= \left\langle \text{Hess f}, df\otimes df\right\rangle V^2|df|^4\left\langle df, \frac{dV}{V}\right\rangle.
\end{aligned}
$$

For P_4 we do a simple redistribution

$$
\begin{aligned}
P_4 &= \nabla_i f \nabla_j f \left(2 \nabla_i f \nabla_j V\right) \frac{1}{V} \nabla_i f \nabla_j f \left(2 \nabla_i f \nabla_j V\right) \frac{1}{V}, \\
&= 2|df|^2 \left\langle df, \frac{dV}{V} \right\rangle 2|df|^2 \left\langle df, \frac{dV}{V} \right\rangle, \\
&= 4|df|^4 \left\langle df, \frac{dV}{V} \right\rangle^2 .
\end{aligned}
$$

For P_5 we do a simple redistribution and split the product into two halfs to match one of the components for C final answer

$$
\begin{aligned}
P_5 + P_9 &= \nabla_i f \nabla_j f V \left\langle df, dV \right\rangle \nabla_i f \nabla_j f \nabla_i f \nabla_j f \left(2 \nabla_i f \nabla_j V\right) \frac{1}{V} \\
&+ \nabla_i f \nabla_j f V \left\langle df, dV \right\rangle \nabla_i f \nabla_j f \nabla_i f \nabla_j f \left(2 \nabla_i f \nabla_j V\right) \frac{1}{V}, \\
&= 4|df|^2 \left\langle df, \frac{dV}{V} \right\rangle V^2 |df|^2 |df|^2 \left\langle df, \frac{dV}{V} \right\rangle, \\
&= 2|df|^2 \left\langle df, \frac{dV}{V} \right\rangle V^2 |df|^2 |df|^2 \left\langle df, \frac{dV}{V} \right\rangle \\
&+ 2|df|^2 \left\langle df, \frac{dV}{V} \right\rangle V^2 |df|^2 |df|^2 \left\langle df, \frac{dV}{V} \right\rangle .
\end{aligned}
$$

For P_6 we do a simple redistribution

$$
\begin{aligned}
P_6 &= \nabla_i f \nabla_j f V \left\langle df, dV \right\rangle \nabla_i f \nabla_j f \nabla_i f \nabla_j f V \left\langle df, dV \right\rangle \nabla_i f \nabla_j f, \\
&= V^2 |df|^2 |df|^2 \left\langle df, \frac{dV}{V} \right\rangle V^2 |df|^2 |df|^2 \left\langle df, \frac{dV}{V} \right\rangle, \\
&= V^4 |df|^8 \left\langle df, \frac{dV}{V} \right\rangle^2 .
\end{aligned}
$$

For P_7 we do a simple redistribution

$$
\begin{aligned}
P_7 &= \nabla_i f \nabla_j f \left(2 \nabla_i f \nabla_j V\right) \frac{1}{V} \nabla_i f \nabla_j f \nabla_i \nabla_j f, \\
&= 2|df|^2 \left\langle df, \frac{dV}{V} \right\rangle \nabla_i f \nabla_j f \nabla_i \nabla_j f, \\
&= 2|df|^2 \left\langle df, \frac{dV}{V} \right\rangle \left\langle \text{Hess f}, df \otimes df \right\rangle .
\end{aligned}
$$

P_8 will be distributed and use (4.29) as substitution

$$
\begin{aligned}
P_8 &= \underbrace{\nabla_i f \nabla_j f V \left\langle df, dV \right\rangle \nabla_i f \nabla_j f \nabla_i f \nabla_j f \nabla_i \nabla_j f}_{\nabla_i = \text{df and distributing in } \frac{V}{V}}, \\
&= \underbrace{V^2 |df|^2 |df|^2 \left\langle df, \frac{dV}{V} \right\rangle \nabla_i f \nabla_j f \nabla_i \nabla_j f}_{\text{using the definition of Hess}}, \\
&= V^2 |df|^4 \left\langle df, \frac{dV}{V} \right\rangle \left\langle \text{Hess f}, df \otimes df \right\rangle .
\end{aligned}
$$

Going back to (5.14) and substituting in $P_1 - P_8$

$$
\begin{aligned}
C = \frac{V^2}{1+V^2|df|^2} \Bigg[& \underbrace{\frac{V^4 \langle \text{Hess f}, df \otimes df \rangle^2}{(1+V^2|df|^2)^2}}_{P_1} + \underbrace{\frac{V^4 2|df|^2 \langle \text{Hess f}, df \otimes df \rangle \left\langle df, \frac{dV}{V} \right\rangle}{(1+V^2|df|^2)^2}}_{P_2} \\
& + \underbrace{\frac{V^4 \langle \text{Hess f}, df \otimes df \rangle V^2 |df|^4 \left\langle df, \frac{dV}{V} \right\rangle}{(1+V^2|df|^2)^2}}_{P_3} + \underbrace{\frac{4V^4|df|^4 \left\langle df, \frac{dV}{V} \right\rangle^2}{(1+V^2|df|^2)^2}}_{P_4} \\
& + \underbrace{\frac{V^4 4|df|^2 \left\langle df, \frac{dV}{V} \right\rangle V^2|df|^2|df|^2 \left\langle df, \frac{dV}{V} \right\rangle}{(1+V^2|df|^2)^2}}_{P_5} + \underbrace{\frac{V^4V^4|df|^8 \left\langle df, \frac{dV}{V} \right\rangle^2}{(1+V^2|df|^2)^2}}_{P_6} \\
& + \underbrace{\frac{V^4 2|df|^2 \left\langle df, \frac{dV}{V} \right\rangle \langle \text{Hess f}, df \otimes df \rangle}{(1+V^2|df|^2)^2}}_{P_7} + \underbrace{\frac{V^4V^2|df|^4 \left\langle df, \frac{dV}{V} \right\rangle \langle \text{Hess f}, df \otimes df \rangle}{(1+V^2|df|^2)^2}}_{P_8} \Bigg],
\end{aligned}
\tag{5.15}
$$

putting all the components together and distributing $\frac{V^4}{1+V^2|df|^2}$ adding the components together

$$
P_1 : \frac{V^4 \langle \text{Hess f}, df \otimes df \rangle^2}{(1+V^2|df|^2)^2} \to \left(\frac{V^2 \langle \text{Hess f}, df \otimes df \rangle}{1+V^2|df|^2} \right)^2,
$$

$$
\begin{aligned}
P_2 + P_7 : & \frac{V^4 2|df|^2 \langle \text{Hess f}, df \otimes df \rangle \left\langle df, \frac{dV}{V} \right\rangle}{(1+V^2|df|^2)^2} + \frac{V^4 2|df|^2 \left\langle df, \frac{dV}{V} \right\rangle \langle \text{Hess f}, df \otimes df \rangle}{(1+V^2|df|^2)^2} \\
= & \frac{V^4 4|df|^2 \langle \text{Hess f}, df \otimes df \rangle \left\langle df, \frac{dV}{V} \right\rangle}{(1+V^2|df|^2)^2},
\end{aligned}
$$

$$
\begin{aligned}
P_3 + P_8 : & \frac{V^4 \langle \text{Hess f}, df \otimes df \rangle V^2|df|^4 \left\langle df, \frac{dV}{V} \right\rangle}{(1+V^2|df|^2)^2} + \frac{V^4V^2|df|^4 \left\langle df, \frac{dV}{V} \right\rangle \langle \text{Hess f}, df \otimes df \rangle}{(1+V^2|df|^2)^2} \\
= & \frac{2V^4V^2|df|^4 \left\langle df, \frac{dV}{V} \right\rangle \langle \text{Hess f}, df \otimes df \rangle}{(1+V^2|df|^2)^2},
\end{aligned}
$$

$$
P_4 : \frac{4V^4|df|^4 \left\langle df, \frac{dV}{V} \right\rangle^2}{(1+V^2|df|^2)^2} \to \frac{\left(2V^2|df|^2 \left\langle df, \frac{dV}{V} \right\rangle \right)^2}{(1+V^2|df|^2)^2},
$$

$$
P_6 : \frac{V^4V^4|df|^8 \left\langle df, \frac{dV}{V} \right\rangle^2}{(1+V^2|df|^2)^2} \to \left(\frac{V^4|df|^4 \left\langle df, \frac{dV}{V} \right\rangle}{1+V^2|df|^2} \right)^2.
$$

With these modifications we see that C equals

$$
\begin{aligned}
C =& \frac{V^2}{(1+V^2|df|^2)^2}\Bigg[\underbrace{\left(\frac{V^2\langle \text{Hess f}, df\otimes df\rangle}{1+V^2|df|^2}\right)^2}_{P_1} + \underbrace{\frac{V^4\langle \text{Hess f}, df\otimes df\rangle 4|df|^2\left\langle df, \frac{dV}{V}\right\rangle}{(1+V^2|df|^2)^2}}_{P_2+P_7}\\
&+ \underbrace{\frac{2V^4\langle \text{Hess f}, df\otimes df\rangle V^2|df|^4\left\langle df, \frac{dV}{V}\right\rangle}{(1+V^2|df|^2)^2}}_{P_3+P_8} + \underbrace{\frac{\left(2|df|^2\left\langle df, \frac{dV}{V}\right\rangle V^2\right)^2}{\left(1+V^2\,(df)^2\right)^2}}_{P_4}\\
&+ \underbrace{\frac{4V^4|df|^2\left\langle df, \frac{dV}{V}\right\rangle V^2|df|^4\left\langle df, \frac{dV}{V}\right\rangle}{\left(1+V^2\,(df)^2\right)}}_{P_5} + \underbrace{\frac{\left(V^4|df|^4\left\langle df, \frac{dV}{V}\right\rangle\right)^2}{\left(1+V^2\,(df)^2\right)^2}\Bigg]}_{P_6}.
\end{aligned}
\tag{5.16}
$$

We can simplify this further as

$$
C = \frac{V^2}{1+V^2|df|^2}\left[\frac{V^2\langle \text{Hess f}, df\otimes df\rangle}{1+V^2|df|^2} + \left(1+\frac{1}{1+V^2|df|^2}\right)V^2|df|^2\left\langle df, \frac{dV}{V}\right\rangle\right]^2. \tag{5.17}
$$

Proof. Notice that working backwards from (5.17) leads to

$$
\begin{aligned}
C =&\frac{V^2}{1+V^2|df|^2}\left[\frac{V^2\left\langle \text{Hess f}, df\otimes df\right\rangle}{1+V^2|df|^2}+\left(1+\frac{1}{1+V^2|df|^2}\right)V^2|df|^2\left\langle df,\frac{dV}{V}\right\rangle\right]^2,\\
=&\frac{V^2}{1+V^2|df|^2}\left[\frac{V^2\left\langle \text{Hess f}, df\otimes df\right\rangle}{1+V^2|df|^2}+\frac{\left(2+V^2|df|^2\right)|df|^2\left\langle df,\frac{dV}{V}\right\rangle}{1+V^2|df|^2}\right]^2,\\
=&\frac{V^2}{(1+V^2|df|^2)}\left[\frac{V^2\left\langle \text{Hess f}, df\otimes df\right\rangle}{1+V^2|df|^2}+\frac{2|df|^2\left\langle df,\frac{dV}{V}\right\rangle V^2}{1+V^2|df|^2},\right.\\
&\left.+\frac{V^2|df|^4\left\langle df,\frac{dV}{V}\right\rangle V^2}{1+V^2|df|^2}\right]\left[\frac{V^2\left\langle \text{Hess f}, df\otimes df\right\rangle}{1+V^2|df|^2}+\frac{2|df|^2\left\langle df,\frac{dV}{V}\right\rangle V^2}{1+V^2|df|^2}+\frac{V^2|df|^4\left\langle df,\frac{dV}{V}\right\rangle V^2}{1+V^2|df|^2}\right],\\
=&\frac{V^2}{(1+V^2|df|^2)^2}\left[\underbrace{\left(\frac{V^2\left\langle \text{Hess f}, df\otimes df\right\rangle}{1+V^2|df|^2}\right)^2}_{P_1}+\underbrace{\frac{V^4\left\langle \text{Hess f}, df\otimes df\right\rangle 4|df|^2\left\langle df,\frac{dV}{V}\right\rangle}{(1+V^2|df|^2)^2}}_{P_2+P_7}\right.\\
&+\underbrace{\frac{2V^4\left\langle \text{Hess f}, df\otimes df\right\rangle V^2|df|^4\left\langle df,\frac{dV}{V}\right\rangle}{(1+V^2|df|^2)^2}}_{P_3+P_8}+\underbrace{\frac{\left(2|df|^2\left\langle df,\frac{dV}{V}\right\rangle V^2\right)^2}{\left(1+V^2\left(df\right)^2\right)^2}}_{P_4}\\
&\left.+\underbrace{\frac{4V^4|df|^2\left\langle df,\frac{dV}{V}\right\rangle V^2|df|^4\left\langle df,\frac{dV}{V}\right\rangle}{\left(1+V^2\left(df\right)^2\right)}}_{P_5}+\underbrace{\frac{\left(V^4|df|^4\left\langle df,\frac{dV}{V}\right\rangle\right)^2}{\left(1+V^2\left(df\right)^2\right)^2}}_{P_6}\right],
\end{aligned}
$$

which is identical to the (5.16), thus the previous calculation can be simplified as

$$
C=\frac{V^2}{1+V^2|df|^2}\left[\frac{V^2\left\langle \text{Hess f}, df\otimes df\right\rangle}{1+V^2|df|^2}+\left(1+\frac{1}{1+V^2|df|^2}\right)V^2|df|^2\left\langle df,\frac{dV}{V}\right\rangle\right]^2.
$$

□

Putting A+B+C together we get $|\overline{S}|_g^2$ □

The mean curvature and second fundamental form of a graph serves in the following equation for Σ scalar curvature

$$\begin{aligned}\overline{H}^2-|\overline{S}|_g^2=&\frac{V^2}{1+V^2|df|^2}\Bigg[(\Delta f)^2-|\mathrm{Hess\ f}|^2+\frac{2V}{1+V^2|df|^2}\left(|\mathrm{Hess\ f}\,(\nabla f,\cdot)\,|^2-\Delta f\left\langle \mathrm{Hess\ f}, df\otimes df\right\rangle\right)\\&+\left(2+\frac{2}{1+V^2|df|^2}\right)\Delta f\left\langle df,\frac{dV}{V}\right\rangle-\frac{2V^2}{1+V^2|df|^2}\left\langle \mathrm{Hess\ f}, df\otimes df\right\rangle\left\langle df,\frac{dV}{V}\right\rangle\\&+\frac{2}{1+V^2|df|^2}\left\langle df,\frac{dV}{V}\right\rangle^2-\frac{2}{1+V^2|df|^2}|df|^2|\frac{dV}{V}|^2-\frac{4}{1+V^2|df|^2}\left\langle \mathrm{Hess\ f}, df\otimes df\right\rangle\Bigg].\end{aligned}$$

Proof. Using Lemma (5.3) and Lemma (5.4), we take the trace of the Gauss equation for Σ (3.17), and arrive at the following curvature R_g of $\overline{H}^2-|\overline{S}|_g^2$

$$\begin{aligned}&=\frac{V^2}{1+V^2|df|^2}\left[\Delta f-\frac{V^2\left\langle \mathrm{Hess\ f}, df\otimes df\right\rangle}{1+V^2|df|^2}+\left(1+\frac{1}{1+V^2|df|^2}\right)\left\langle df,\frac{dV}{V}\right\rangle\right]^2\\&-\frac{V^2}{1+V^2|df|^2}\Bigg[|\mathrm{Hess\ f}|^2+2|df|^2|\frac{dV}{|V|^2}+2\left\langle df,\frac{dV}{V}\right\rangle^2+V^4|df|^4\left\langle df,\frac{dV}{V}\right\rangle^2\\&+4\left\langle \mathrm{Hess\ f}, df\otimes df\right\rangle+2V^2\left\langle df,\frac{dV}{V}\right\rangle\left\langle \mathrm{Hess\ f}, df\otimes df\right\rangle+4V^2|df|^2\left\langle df,\frac{dV}{V}\right\rangle^2\Bigg]\\&+\frac{2V^4}{(1+V^2|df|^2)^2}\Bigg[|\mathrm{Hess\ f}\,(\nabla f,\cdot)\,|^2+\left(1+V^2|df|^2\right)^2\left\langle df,\frac{dV}{V}\right\rangle^2|df|^2\\&+|df|^4|\frac{dV}{V}|^2+2\left(1+V^2|df|^2\right)\mathrm{Hess\ f}\,(\nabla f,\nabla f)\left\langle df,\frac{dV}{V}\right\rangle\\&+2|df|^2\left\langle \mathrm{Hess\ f},\nabla f\otimes\frac{\nabla V}{V}\right\rangle+2\left(1+V^2|df|^2\right)|df|^2\left\langle df,\frac{dV}{V}\right\rangle^2\Bigg]\\&-\frac{V^2}{1+V^2|df|^2}\left[\frac{V^2\left\langle \mathrm{Hess\ f}, df\otimes df\right\rangle}{1+V^2|df|^2}+\left(1+\frac{1}{1+V^2|df|^2}\right)V^2|df|^2\left\langle df,\frac{dV}{V}\right\rangle\right]^2,\end{aligned}$$

factor out $\dfrac{V^2}{1+V^2|df|^2}$ and distribute $\left[\Delta f - \dfrac{V^2\left\langle \text{Hess f}, df\otimes df\right\rangle}{1+V^2|df|^2} + \left(1+\dfrac{1}{1+V^2|df|^2}\right)\left\langle df, \dfrac{dV}{V}\right\rangle\right]^2$

$$
\begin{aligned}
&= \frac{V^2}{1+V^2|df|^2}\Bigg(\Bigg[\underbrace{\Delta f\Delta f}_{V_1} - \underbrace{\frac{2\Delta f V^2\left\langle \text{Hess f}, df\otimes df\right\rangle}{1+V^2|df|^2}}_{V_2} + \underbrace{2\Delta f\left(1+\frac{1}{1+V^2|df|^2}\right)\left\langle df, \frac{dV}{V}\right\rangle}_{V_3} \\
&+ \underbrace{\left(\frac{V^2\left\langle \text{Hess f}, df\otimes df\right\rangle}{1+V^2|df|^2}\right)^2}_{V_4} - \underbrace{\frac{2V^2\left\langle \text{Hess f}, df\otimes df\right\rangle}{1+V^2|df|^2}\left(1+\frac{1}{1+V^2|df|^2}\right)\left\langle df, \frac{dV}{V}\right\rangle}_{V_5} \\
&+ \underbrace{\left(\left(1+\frac{1}{1+V^2|df|^2}\right)\left\langle df, \frac{dV}{V}\right\rangle\right)^2}_{V_6}\Bigg] \\
&- \Bigg[\underbrace{|\text{Hess f}|^2}_{V_{10}} + \underbrace{2|df|^2\left|\frac{dV}{V}\right|^2}_{V_{11}} + \underbrace{2\left\langle df, \frac{dV}{V}\right\rangle^2}_{V_{12}} + \underbrace{V^4|df|^4\left\langle df, \frac{dV}{V}\right\rangle^2}_{V_{13}} \\
&+ \underbrace{4\left\langle \text{Hess f}, df\otimes\frac{dV}{V}\right\rangle}_{V_{14}} + \underbrace{2V^2\left\langle df, \frac{dV}{V}\right\rangle\left\langle \text{Hess f}, df\otimes df\right\rangle}_{V_{15}} + \underbrace{4V^2|df|^2\left\langle df, \frac{dV}{V}\right\rangle^2}_{V_{16}}\Bigg] \\
&+ \frac{2V^2}{1+V^2|df|^2}\Bigg[\underbrace{|\text{Hess f}(\nabla f,\cdot)|^2}_{V_{17}} + \underbrace{\left(1+V^2|df|^2\right)^2\left\langle df, \frac{dV}{V}\right\rangle^2|df|^2}_{V_{18}} + \underbrace{|df|^4\left|\frac{dV}{V}\right|^2}_{V_{19}} \\
&+ \underbrace{2\left(1+V^2|df|^2\right)\text{Hess f}(\nabla f,\nabla f)\left\langle df, \frac{dV}{V}\right\rangle}_{V_{20}} \\
&+ \underbrace{2|df|^2\left\langle \text{Hess f}, \nabla\otimes\frac{\nabla V}{V}\right\rangle}_{V_{21}} + \underbrace{2\left(1+V^2|df|^2\right)|df|^2\left\langle df, \frac{dV}{V}\right\rangle^2}_{V_{22}}\Bigg] \\
&- \Bigg[\underbrace{\left(\frac{V^2\left\langle \text{Hess f}, df\otimes df\right\rangle}{1+V^2|df|^2}\right)^2}_{V_7} + \underbrace{\frac{2V^4\left\langle \text{Hess f}, df\otimes df\right\rangle|df|^2\left\langle df, \frac{dV}{V}\right\rangle}{1+V^2|df|^2}\left(1+\frac{1}{1+V^2|df|^2}\right)}_{V_8} \\
&+ \underbrace{\left(\left(1+\frac{1}{1+V^2|df|^2}\right)V^2|df|^2\left\langle df, \frac{dV}{V}\right\rangle\right)^2}_{V_9}\Bigg].
\end{aligned}
$$

For V_{12} and V_{18} and V_{13},

$$V_{12}+V_{18}=\underbrace{-2\left\langle df,\frac{dV}{V}\right\rangle^2}_{V_{12}}+\underbrace{\frac{2V^2}{(1+V^2|df|^2)}\left(\left(1+V^2|df|^2\right)^2|df|^2\left\langle df,\frac{dV}{V}\right\rangle^2\right)}_{V_{18}},$$

$$V_{12}+V_{18}=-2\left\langle df,\frac{dV}{V}\right\rangle^2+2V^2\left(\left(1+V^2|df|^2\right)|df|^2\left\langle df,\frac{dV}{V}\right\rangle^2\right),$$

$$V_{12}+V_{18}=-2\left\langle df,\frac{dV}{V}\right\rangle^2+2V^2|df|^2\left\langle df,\frac{dV}{V}\right\rangle^2+2V^4|df|^4\left\langle df,\frac{dV}{V}\right\rangle^2,$$

$$V_{12}+V_{18}+V_{13}=-2\left\langle df,\frac{dV}{V}\right\rangle^2+2V^2|df|^2\left\langle df,\frac{dV}{V}\right\rangle^2+2V^4|df|^4\left\langle df,\frac{dV}{V}\right\rangle^2-\underbrace{V^4|df|^4\left\langle df,\frac{dV}{V}\right\rangle}_{V_{13}},$$

$$V_{12}+V_{18}+V_{13}=-2\left\langle df,\frac{dV}{V}\right\rangle^2+2V^2|df|^2\left\langle df,\frac{dV}{V}\right\rangle^2+V^4|df|^4\left\langle df,\frac{dV}{V}\right\rangle^2,$$

$$V_{12}+V_{18}+V_{13}=\left(-2+2V^2|df|^2+V^4|df|^4\right)\left\langle df,\frac{dV}{V}\right\rangle^2,$$

For V_{14} and V_{21},

$$V_{14}+V_{21}=\underbrace{-4\left\langle \text{Hess f}, df\otimes\frac{dV}{V}\right\rangle}_{V_{14}}+\underbrace{\frac{2V^2}{1+V^2|df|^2}\left(2|df|^2\left\langle \text{Hess f},\nabla f\otimes\frac{\nabla V}{V}\right\rangle\right)}_{V_{21}},$$

rewriting $-4\left\langle \text{Hess f}, df\otimes\frac{dV}{V}\right\rangle=-4\left\langle \text{Hess f}, df\otimes\frac{dV}{V}\right\rangle\frac{1+V^2|df|^2}{1+V^2|df|^2}$ we see that

$$\begin{aligned}
V_{14}+V_{21}=&\left(\frac{-4\left\langle \text{Hess f}, df\otimes\frac{dV}{V}\right\rangle-4V^2|df|^2\left\langle \text{Hess f}, df\otimes\frac{dV}{V}\right\rangle}{1+V^2|df|^2}\right.\\
&\left.+\frac{2V^2}{1+V^2|df|^2}\left(2|df|^2\left\langle \text{Hess f},\nabla f\otimes\frac{\nabla V}{V}\right\rangle\right)\right),\\
=&\left(\frac{-4\left\langle \text{Hess f}, df\otimes\frac{\nabla V}{V}\right\rangle}{1+V^2|df|^2}-\frac{4V^2|df|^2\left\langle \text{Hess f}, df\otimes\frac{\nabla V}{V}\right\rangle}{1+V^2|df|^2}\right.\\
&\left.+\frac{2V^2}{1+V^2|df|^2}\left(2|df|^2\left\langle \text{Hess f},\nabla f\otimes\frac{\nabla V}{V}\right\rangle\right)\right),\\
=&\frac{-4\left\langle \text{Hess f}, df\otimes\frac{\nabla V}{V}\right\rangle}{1+V^2|df|^2}-\frac{4V^2|df|^2\left\langle \text{Hess f}, df\otimes\frac{\nabla V}{V}\right\rangle}{1+V^2|df|^2}+\frac{4V^2|df|^2\left\langle \text{Hess f},\nabla f\otimes\frac{\nabla V}{V}\right\rangle}{1+V^2},\\
=&\frac{-4\left\langle \text{Hess f},\nabla f\otimes\frac{\nabla V}{V}\right\rangle}{1+V^2|df|^2}.
\end{aligned}$$

For V_{20} and V_{15}

$$V_{20} = \underbrace{\frac{2V^2}{1+V^2|df|^2}\left(2\left(1+V^2|df|^2\right)\text{Hess f}\left(\nabla f,\nabla f\right)\left\langle df,\frac{dV}{V}\right\rangle\right)}_{V_{20}},$$

$$\begin{aligned}
V_{20} =& \underbrace{\frac{4V^2\text{Hess f}\left(\nabla f,\nabla f\right)\left\langle df,\frac{dV}{V}\right\rangle}{1+V^2|df|^2} + \frac{4V^4|df|^2\text{Hess f}\left(\nabla f,\nabla f\right)\left\langle df,\frac{dV}{V}\right\rangle}{1+V^2|df|^2}}_{\text{Distributing } \frac{2V^2}{1+V^2|df|^2}}, \\
=& 4V^2\text{Hess f}\left(\nabla f,\nabla f\right)\left\langle df,\frac{dV}{V}\right\rangle\frac{\left(1+V^2|df|^2\right)}{1+V^2|df|^2}, \\
=& 4V^2\text{Hess f}\left(\nabla f,\nabla f\right)\left\langle df,\frac{dV}{V}\right\rangle, \\
=& 2V^2\text{Hess f}\left(\nabla f,\nabla f\right)\left\langle df,\frac{dV}{V}\right\rangle \\
& + 2V^2\text{Hess f}\left(\nabla f,\nabla f\right)\left\langle df,\frac{dV}{V}\right\rangle, \\
V_{20}+V_{15} =& 2V^2\text{Hess f}\left(\nabla f,\nabla f\right)\left\langle df,\frac{dV}{V}\right\rangle \\
& + 2V^2\text{Hess f}\left(\nabla f,\nabla f\right)\left\langle df,\frac{dV}{V}\right\rangle \\
& \underbrace{- 2V^2\text{Hess f}\left(\nabla f,\nabla f\right)\left\langle df,\frac{dV}{V}\right\rangle}_{V_{15}}, \\
V_{20}+V_{15} =& 2V^2\text{Hess f}\left(\nabla f,\nabla f\right)\left\langle df,\frac{dV}{V}\right\rangle.
\end{aligned}$$

For V_{11} and V_{19}

$$V_{11} = \underbrace{-2|df|^2|\frac{dV}{V}|^2}_{V_{11}} = \frac{-2|df|^2|\frac{dV}{V}|^2 - 2V^2|df|^4|\frac{dV}{V}|^2}{1+V^2|df|^2},$$

$$\begin{aligned}
V_{11}+V_{19} =& \frac{-2|df|^2|\frac{dV}{V}|^2}{1+V^2|df|^2} - \frac{2V^2|df|^4|\frac{dV}{V}|^2}{1+V^2|df|^2} + \underbrace{\frac{2V^2}{1+V^2|df|^2}\left(|df|^4|\frac{dV}{V}|^2\right)}_{V_{19}}, \\
V_{11}+V_{19} =& \frac{-2|df|^2|\frac{dV}{V}|^2}{1+V^2|df|^2} - \frac{2V^2|df|^4|\frac{dV}{V}|^2}{1+V^2|df|^2} + \frac{2V^2|df|^4|\frac{dV}{V}|^2}{1+V^2|df|^2}, \\
V_{11}+V_{19} =& \frac{-2|df|^2|\frac{dV}{V}|^2}{1+V^2|df|^2}.
\end{aligned}$$

Now lets combine and rewrite more things together

$$V_2 + V_{17} = \underbrace{\frac{-2\Delta f V^2 \langle \text{Hess f}, df \otimes df \rangle}{1 + V^2|df|^2}}_{V_2} + \underbrace{\frac{2V^2 \left(|\text{Hess f}(\nabla f, \cdot)|^2 \right)}{1 + V^2|df|^2}}_{V_{17}},$$

$$V_2 + V_{17} = \frac{2V^2}{1 + V^2|df|^2} \left(|\text{Hess f}(\nabla f, \cdot)|^2 - \Delta f \langle \text{Hess f}, df \otimes df \rangle \right).$$

Rewriting V_3

$$V_3 = \underbrace{2\Delta f \left(1 + \frac{1}{1 + V^2|df|^2} \right) \left\langle df, \frac{dV}{V} \right\rangle}_{V_3},$$

$$V_3 = \left(2 + \frac{2}{1 + V^2|df|^2} \right) \Delta f \left\langle df, \frac{dV}{V} \right\rangle.$$

For $V_5 + V_8 + V_{20} + V_{15}$

$$V_5 = \underbrace{\frac{-2V^2 \langle \text{Hess f}, df \otimes df \rangle}{1 + V^2|df|^2} \left(1 + \frac{1}{1 + V^2|df|^2} \right) \left\langle df, \frac{dV}{V} \right\rangle}_{V_5},$$

$$\begin{aligned} V_5 + V_8 = & \frac{-2V^2 \langle \text{Hess f}, df \otimes df \rangle \left\langle df, \frac{dV}{V} \right\rangle}{1 + V^2|df|^2} - \frac{2V^2 \langle \text{Hess f}, df \otimes df \rangle \left\langle df, \frac{dV}{V} \right\rangle}{(1 + V^2|df|^2)^2} \\ & \underbrace{- \frac{2V^4 \langle \text{Hess f}, df \otimes df \rangle |df|^2 \left\langle df, \frac{dV}{V} \right\rangle}{1 + V^2|df|^2} \left(1 + \frac{1}{1 + V^2|df|^2} \right)}_{V_8}, \end{aligned}$$

$$\begin{aligned} V_5 + V_8 = & \frac{-2V^2 \langle \text{Hess f}, df \otimes df \rangle \left\langle df, \frac{dV}{V} \right\rangle}{1 + V^2|df|^2} - \frac{2V^2 \langle \text{Hess f}, df \otimes df \rangle \left\langle df, \frac{dV}{V} \right\rangle}{(1 + V^2|df|^2)^2} \\ & - \frac{2V^4 \langle \text{Hess f}, df \otimes df \rangle |df|^2 \left\langle df, \frac{dV}{V} \right\rangle}{1 + V^2|df|^2} - \frac{2V^4 \langle \text{Hess f}, df \otimes df \rangle |df|^2 \left\langle df, \frac{dV}{V} \right\rangle}{(1 + V^2|df|^2)^2}, \end{aligned}$$

$$\begin{aligned} V_5 + V_8 + V_{20} + V_{15} = & \frac{-2V^2 \langle \text{Hess f}, df \otimes df \rangle \left\langle df, \frac{dV}{V} \right\rangle}{1 + V^2|df|^2} - \frac{2V^2 \langle \text{Hess f}, df \otimes df \rangle \left\langle df, \frac{dV}{V} \right\rangle}{(1 + V^2|df|^2)^2} \\ & - \frac{2V^4 \langle \text{Hess f}, df \otimes df \rangle |df|^2 \left\langle df, \frac{dV}{V} \right\rangle}{1 + V^2|df|^2} - \frac{2V^4 \langle \text{Hess f}, df \otimes df \rangle |df|^2 \left\langle df, \frac{dV}{V} \right\rangle}{(1 + V^2|df|^2)^2} \\ & + \underbrace{2V^2 \text{Hess f}(\nabla f, \nabla f) \left\langle df, \frac{dV}{V} \right\rangle}_{V_{20} + V_{15}}. \end{aligned}$$

Rewrite $2V^2\text{Hess f}\left(\nabla f,\nabla f\right)\left\langle df,\frac{dV}{V}\right\rangle = 2V^2\text{Hess f}\left(\nabla f,\nabla f\right)\left\langle df,\frac{dV}{V}\right\rangle \times \frac{1+V^2|df|^2}{1+V^2|df|^2}$

$$
\begin{aligned}
V_5+V_8+V_{20}+V_{15} =& \frac{-2V^2\left\langle \text{Hess f}, df\otimes df\right\rangle \left\langle df,\frac{dV}{V}\right\rangle}{1+V^2|df|^2} - \frac{2V^4\left\langle \text{Hess f}, df\otimes df\right\rangle |df|^2 \left\langle df,\frac{dV}{V}\right\rangle}{\left(1+V^2|df|^2\right)^2} \\
&- \frac{2V^4\left\langle \text{Hess f}, df\otimes df\right\rangle |df|^2 \left\langle df,\frac{dV}{V}\right\rangle}{1+V^2|df|^2} \\
&+ \frac{2V^2\left\langle df,\frac{dV}{V}\right\rangle \left\langle \text{Hess f}, df\otimes df\right\rangle + 2V^4|df|^2\left\langle df,\frac{dV}{V}\right\rangle \left\langle \text{Hess f}, df\otimes df\right\rangle}{1+V^2|df|^2} \\
&- \frac{2V^2\left\langle \text{Hess f}, df\otimes df\right\rangle \left\langle df,\frac{dV}{V}\right\rangle}{1+V^2|df|^2}, \\
V_5+V_8+V_{20}+V_{15} =& \frac{-2V^2\left\langle \text{Hess f}, df\otimes df\right\rangle \left\langle df,\frac{dV}{V}\right\rangle}{1+V^2|df|^2} - \frac{2V^4\left\langle \text{Hess f}, df\otimes df\right\rangle |df|^2 \left\langle df,\frac{dV}{V}\right\rangle}{\left(1+V^2|df|^2\right)^2} \\
&- \frac{2V^4\left\langle \text{Hess f}, df\otimes df\right\rangle |df|^2 \left\langle df,\frac{dV}{V}\right\rangle}{1+V^2|df|^2} \\
&+ \frac{2V^2\left\langle df,\frac{dV}{V}\right\rangle \left\langle \text{Hess f}, df\otimes df\right\rangle}{1+V^2|df|^2} + \frac{2V^4|df|^2\left\langle df,\frac{dV}{V}\right\rangle \left\langle \text{Hess f}, df\otimes df\right\rangle}{1+V^2|df|^2} \\
&- \frac{2V^2\left\langle \text{Hess f}, df\otimes df\right\rangle \left\langle df,\frac{dV}{V}\right\rangle}{\left(1+V^2|df|^2\right)^2}, \\
V_5+V_8+V_{20}+V_{15} =& -\frac{2V^2\left\langle \text{Hess f}, df\otimes df\right\rangle \left\langle df,\frac{dV}{V}\right\rangle}{\left(1+V^2|df|^2\right)^2} - \frac{2V^4\left\langle \text{Hess f}, df\otimes df\right\rangle |df|^2 \left\langle df,\frac{dV}{V}\right\rangle}{\left(1+V^2|df|^2\right)^2}, \\
V_5+V_8+V_{20}+V_{15} =& -2V^2\left\langle \text{Hess f}, df\otimes df\right\rangle \left\langle df,\frac{dV}{V}\right\rangle \frac{\left(1+V^2|df|^2\right)}{\left(1+V^2|df|^2\right)^2}, \\
V_5+V_8+V_{20}+V_{15} =& \frac{-2V^2\left\langle \text{Hess f}, df\otimes df\right\rangle \left\langle df,\frac{dV}{V}\right\rangle}{1+V^2|df|^2}.
\end{aligned}
$$

Also see that

$$V_{12}+V_{18}+V_{13} = \left(-2+2V^2|df|^2+V^4|df|^4\right)\left\langle df,\frac{dV}{V}\right\rangle^2$$

distribute$\left\langle df, \frac{dV}{V}\right\rangle^2$ and multiply by $\frac{1+V^2|df|^2}{1+V^2|df|^2}$. From there, the calculation $V_{12}+V_{18}+V_{13}$ will be temporarily relabeled as O to make the following calculations clearer.

$$\begin{aligned} V_{12}+V_{18}+V_{13} = {} & \underbrace{\frac{-2\left\langle df, \frac{dV}{V}\right\rangle^2}{1+V^2|df|^2}}_{O_1} - \underbrace{\frac{2V^2|df|^2\left\langle df, \frac{dV}{V}\right\rangle^2}{1+V^2|df|^2}}_{O_2} \\ & + \underbrace{\frac{2V^2|df|^2\left\langle df, \frac{dV}{V}\right\rangle^2}{1+V^2|df|^2}}_{O_3} + \underbrace{\frac{2V^4|df|^4\left\langle df, \frac{dV}{V}\right\rangle^2}{1+V^2|df|^2}}_{O_4} \\ & + \underbrace{\frac{V^4|df|^4\left\langle df, \frac{dV}{V}\right\rangle^2}{1+V^2|df|^2}}_{O_5} + \underbrace{\frac{V^6|df|^6\left\langle df, \frac{dV}{V}\right\rangle^2}{1+V^2|df|^2}}_{O_6}. \end{aligned} \tag{5.18}$$

For V_6 we will rewrite it as

$$V_6 = \underbrace{\left(\left(1+\frac{1}{1+V^2|df|^2}\right)\left\langle df, \frac{dV}{V}\right\rangle\right)^2}_{(V_6)} = \left(\frac{2}{1+V^2|df|^2}+\frac{V^2|df|^2}{1+V^2|df|^2}\right)^2\left\langle df, \frac{dV}{V}\right\rangle^2$$

For which I will temporarily label as K and distribute and rewrite as $K_1+K_2+K_3$

$$\left(\frac{2}{1+V^2|df|^2}+\frac{V^2|df|^2}{1+V^2|df|^2}\right)^2\left\langle df, \frac{dV}{V}\right\rangle^2 = \underbrace{\frac{4\left\langle df, \frac{dV}{V}\right\rangle^2}{(1+V^2|df|^2)^2}}_{K_1} + \underbrace{\frac{4V^2|df|^2}{(1+V^2|df|^2)^2}}_{K_2} + \underbrace{\frac{V^4|df|^4\left\langle df, \frac{dV}{V}\right\rangle^2}{(1+V^2|df|^2)^2}}_{K_3}.$$

Now for V_9 we distribute and rewrite it temporarily with label Q

$$\begin{aligned} -\underbrace{\left(\left(1+\frac{1}{1+V^2|df|^2}\right)V^2|df|^2\left\langle df, \frac{dV}{V}\right\rangle\right)^2}_{V_9} &= -\left(\left(\frac{2+V^2|df|^2}{1+V^2|df|^2}\right)V^2|df|^2\left\langle df, \frac{dV}{V}\right\rangle\right)^2 \\ &= -\left(\frac{2V^2|df|^2\left\langle df, \frac{dV}{V}\right\rangle}{1+V^2|df|^2}+\frac{V^4|df|^4\left\langle df, \frac{dV}{V}\right\rangle}{1+V^2|df|^2}\right)^2. \end{aligned} \tag{5.19}$$

For which we will distribute and rewrite as $Q_1+Q_2+Q_3$

$$Q_1+Q_2+Q_3 = \underbrace{\frac{-4V^4|df|^4\left\langle df, \frac{dV}{V}\right\rangle^2}{(1+V^2|df|)^2}}_{Q_1} - \underbrace{\frac{4V^6|df|^6\left\langle df, \frac{dV}{V}\right\rangle^2}{(1+V^2|df|^2)^2}}_{Q_2} - \underbrace{\left(\frac{V^4|df|^4\left\langle df, \frac{dV}{V}\right\rangle}{1+V^2|df|^2}\right)^2}_{Q_3}.$$

Collectively, $V_{12}+V_{18}+V_{13}+V_9+V_6 = Q_1+Q_2+Q_3 + K_1+K_2+K_3 + O_1+O_2+O_3+O_4+O_5+O_6$

$$Q_2+Q_3 = \underbrace{\frac{-4V^6|df|^6\left\langle df, \dfrac{dV}{V}\right\rangle^2}{(1+V^2|df|^2)^2}}_{Q_2} - \underbrace{\left(\frac{V^4|df|^4\left\langle df, \dfrac{dV}{V}\right\rangle}{1+V^2|df|^2}\right)^2}_{Q_3},$$

$$Q_2+Q_3 = \frac{-4V^6|df|^6\left\langle df, \dfrac{dV}{V}\right\rangle^2}{(1+V^2|df|^2)^2} - \frac{V^8|df|^8\left\langle df, \dfrac{dV}{V}\right\rangle^2}{(1+V^2|df|^2)^2},$$

$$Q_2+Q_3 = \frac{-3V^6|df|^6\left\langle df, \dfrac{dV}{V}\right\rangle^2}{(1+V^2|df|^2)^2} - V^6|df|^6\frac{1+V^2|df|^2}{(1+V^2|df|^2)^2}\left\langle df, \frac{dV}{V}\right\rangle^2,$$

$$Q_2+Q_3+O_6 = \frac{-3V^6|df|^6\left\langle df, \dfrac{dV}{V}\right\rangle^2}{(1+V^2|df|^2)^2} - \frac{V^6|df|^6\left\langle df, \dfrac{dV}{V}\right\rangle^2}{1+V^2|df|^2} + \underbrace{\frac{V^6|df|^6\left\langle df, \dfrac{dV}{V}\right\rangle^2}{(1+V^2|df|^2)^2}}_{O_6},$$

$$Q_2+Q_3+O_6+Q_1+K_3 = \frac{-3V^6|df|^6\left\langle df, \dfrac{dV}{V}\right\rangle^2}{(1+V^2|df|^2)^2} - \underbrace{\frac{4V^4|df|^4\left\langle df, \dfrac{dV}{V}\right\rangle^2}{(1+V^2|df|^2)^2}}_{Q_1} + \underbrace{\frac{V^4|df|^4\left\langle df, \dfrac{dV}{V}\right\rangle^2}{(1+V^2|df|^2)^2}}_{K_3},$$

$$Q_2+Q_3+O_6+Q_1+K_3+O_4+O_5 = \frac{-3V^4|df|^4\left\langle df, \dfrac{dV}{V}\right\rangle^2}{1+V^2|df|^2} + \underbrace{\frac{2V^4|df|^4\left\langle df, \dfrac{dV}{V}\right\rangle^2}{1+V^2|df|^2}}_{O_4} + \underbrace{\frac{V^4|df|^4\left\langle df, \dfrac{dV}{V}\right\rangle^2}{1+V^2|df|^2}}_{O_5},$$

$$
\begin{aligned}
Q_2+Q_3+O_6+Q_1+K_3+O_4+O_5+K_1+O_3+K_2 &= \underbrace{\frac{4\left\langle df,\frac{dV}{V}\right\rangle^2}{\left(1+V^2|df|^2\right)^2}}_{K_1}-\underbrace{\frac{2V^2|df|^2\left\langle df,\frac{dV}{V}\right\rangle^2}{1+V^2|df|^2}}_{O_3}\\
&\quad+\underbrace{\frac{4V^2|df|^2\left\langle df,\frac{dV}{V}\right\rangle^2}{\left(1+V^2|df|^2\right)^2}}_{K_2},\\
Q_2+Q_3+O_6+Q_1+K_3+O_4+O_5+K_1+O_3+K_2 &= \frac{4\left(1+V^2|df|^2\right)\left\langle df,\frac{dV}{V}\right\rangle^2}{\left(1+V^2|df|^2\right)^2}\\
&\quad-\frac{2V^2|df|^2\left\langle df,\frac{dV}{V}\right\rangle^2}{1+V^2|df|^2},\\
Q_2+Q_3+O_6+Q_1+K_3+O_4+O_5+K_1+O_3+K_2+O_2 &= \frac{4\left\langle df,\frac{dV}{V}\right\rangle^2}{1+V^2|df|^2}-\frac{2V^2|df|^2\left\langle df,\frac{dV}{V}\right\rangle^2}{1+V^2|df|^2}\\
&\quad+\underbrace{\frac{2V^2|df|^2\left\langle df,\frac{dV}{V}\right\rangle^2}{1+V^2|df|^2}}_{O_2},\\
Q_2+Q_3+O_6+Q_1+K_3+O_4+O_5+K_1+O_3+K_2+O_2+O_1 &= \frac{4\left\langle df,\frac{dV}{V}\right\rangle^2}{1+V^2|df|^2}-\underbrace{\frac{2\left\langle df,\frac{dV}{V}\right\rangle^2}{1+V^2|df|^2}}_{O_1},\\
Q_2+Q_3+O_6+Q_1+K_3+O_4+O_5+K_1+O_3+K_2+O_2+O_1 &= \frac{2\left\langle df,\frac{dV}{V}\right\rangle^2}{1+V^2|df|^2}.
\end{aligned}
$$

Thus

$$
V_{12}+V_{18}+V_{13}+V_9+V_6=\frac{2\left\langle df,\frac{dV}{V}\right\rangle^2}{1+V^2|df|^2}.
$$

Now for the remaining components we will see that V_{22}, V_{16}, V_4, V_7 cancel out

$$\begin{aligned}
V_{22}+V_{16} =& \frac{2V^2}{1+V^2|df|^2}\left(2\left(1+V^2|df|^2\right)|df|^2\left\langle df, \frac{dV}{V}\right\rangle^2\right) - 4V^2|df|^2\left\langle df, \frac{dV}{V}\right\rangle^2, \\
=& \left(4V^2|df|^2\left\langle df, \frac{dV}{V}\right\rangle^2\right) - 4V^2|df|^2\left\langle df, \frac{dV}{V}\right\rangle^2, \\
=& 0, \\
V_4+V_7 =& \left(\frac{V^2\left\langle \text{Hess f}, df\otimes df\right\rangle}{1+V^2|df|^2}\right)^2 - \left(\frac{V^2\left\langle \text{Hess f}, df\otimes df\right\rangle}{1+V^2|df|^2}\right)^2, \\
=& 0.
\end{aligned}$$

With all the summations in mind, all together $R_g + n(n-1)$ is
$\overline{H}^2 - |\overline{S}|_g^2 =$

$$\begin{aligned}
=& \frac{V^2}{1+V^2|df|^2}\Bigg[\underbrace{(\Delta f)^2}_{V_1} - \underbrace{|\text{Hess f}|^2}_{V_{10}} + \underbrace{\frac{2V}{1+V^2|df|^2}\left(|\text{Hess f}(\nabla f, \cdot)|^2 - \Delta f\left\langle \text{Hess f}, df\otimes df\right\rangle\right)}_{V_2+V_{17}} \\
&+ \underbrace{\left(2+\frac{2}{1+V^2|df|^2}\right)\Delta f\left\langle df, \frac{dV}{V}\right\rangle}_{V_3} - \underbrace{\frac{2V^2}{1+V^2|df|^2}\left\langle \text{Hess f}, df\otimes df\right\rangle\left\langle df, \frac{dV}{V}\right\rangle}_{V_5+V_8+V_{20}+V_{15}} \\
&+ \underbrace{\frac{2}{1+V^2|df|^2}\left\langle df, \frac{dV}{V}\right\rangle^2}_{V_{12}+V_{18}+V_{13}+V_6+V_9} - \underbrace{\frac{2}{1+V^2|df|^2}|df|^2\left|\frac{dV}{V}\right|^2}_{V_{11}+V_{19}} - \underbrace{\frac{4}{1+V^2|df|^2}\left\langle \text{Hess f}, df\otimes\frac{dV}{V}\right\rangle}_{V_{14}+V_{21}}\Bigg].
\end{aligned}$$

□

Lemma 5.5. *We have the following identity*

$$div^b\left[\frac{1}{1+V^2|df|^2}\left(V\ div\ {}^b e - V\ d\ tr^b e - e(\nabla V, \cdot) + \left(tr^b e\right)dV\right)\right] = V\left(\overline{H}^2 - |\overline{S}|_g^2\right) \tag{5.20}$$

where $e = V^2 df\otimes df$.

Proof. Starting with: V $div^b\left(V^2 df\otimes df\right)$- V d $tr^b\left(V^2 df\otimes df\right)$-$V^2 df\otimes df\left(\nabla V, \cdot\right) + \left(tr^b V^2 df\otimes df\right)dV$
We see that $div^b V^2 df\otimes df$ is

$$\begin{aligned}
div^b V^2 df\otimes df =& 2V\nabla V df\otimes df + V^2\nabla f df\otimes df + V^2 df\nabla f df \\
=& 2V\left\langle df, dV\right\rangle df + V^2\nabla\nabla f df + V^2 df\nabla\nabla f, \\
=& 2V\left\langle df, dV\right\rangle df + V^2\Delta f df + V^2\left\langle \text{Hess f}, df\otimes\cdot\right\rangle.
\end{aligned} \tag{5.21}$$

And that $tr^b V^2 df\otimes df = V^2 df\otimes df$.
So
(5.22)

$$\begin{aligned}
V\ \text{div}\ {}^b e - \text{V d}\ tr^b e - e(\nabla V, \cdot) + \left(tr^b e\right)dV =& V\underbrace{\left(2V\left\langle df, dV\right\rangle df + V^2\nabla f df + V^2\left\langle \text{Hess f}, df\otimes\cdot\right\rangle\right)}_{\left(div^b V^2 df\otimes df\right)} \\
&- \text{V d}\ tr^b\left(V^2|df|^2\right) - V^2 df df\left(\nabla V, \cdot\right) + V^2|df|^2 dV, \\
=& 2V^2\left\langle df, dV\right\rangle df + V^3\nabla f df + V^3\left\langle \text{Hess f}, df\otimes\cdot\right\rangle \\
&- \text{V d}\ tr^b\left(V^2|df|^2\right) - V^2\left\langle df, dV\right\rangle df + V^2|df|^2 dV.
\end{aligned}$$

An asymptotic property of $d\ tr^b\left(V^2|df|^2\right)$ appears as follow

$$\begin{aligned}
=&2V^2\left\langle df, dV\right\rangle df + V^3\nabla f df + V^3\left\langle \text{Hess f}, df\otimes\cdot\right\rangle\\
&+\underbrace{dV\left(-2V|df|^2\right) - V^3\nabla\nabla f\nabla f - V^3\nabla f\nabla\nabla f}_{\text{asymptotic property}} - V^2\left\langle df, dV\right\rangle df + V^2|df|^2 dV,\\
=&V^3\left\langle \text{Hess f}, df\otimes\cdot\right\rangle - 2V^3\left\langle \text{Hess f}, df\otimes\cdot\right\rangle + 2V^2\left\langle df, dV\right\rangle df - 2V\left\langle df, dV\right\rangle df\\
&+ V^3\Delta f df - V^2|df|^2 dV + V^2|df|^2 dV\\
&V^3\Delta f df - V^3\left\langle \text{Hess f}, df\otimes\cdot\right\rangle - V^2|df|^2 dV + V^2\left\langle df, dV\right\rangle df.
\end{aligned}$$

With this in mind for $V div^b$e- v d tr^be-e$(\nabla V,\cdot)$+$(tr^b$e)dV let's look back at div^b (V div^be- v d tr^be-e$(\nabla V,\cdot)$+$(tr^b$e)dV) with d Hess f $=$ div^b Hess f df in mind

$$\begin{aligned}
=&div^b\left(V^3\Delta f df - V^3\left\langle \text{Hess f}, df\otimes\cdot\right\rangle - V^2|df|^2 dV + V^2\left\langle df, dV\right\rangle df\right),\\
=&3V^2\Delta f df + V^3 d\Delta f df + V^3\Delta f\nabla\nabla f\\
&- 3V^2\left\langle \text{Hess f}, df\otimes df\right\rangle - V^3\left\langle \text{div}^b\text{ Hess f}, df\right\rangle - V^3\left\langle \text{Hess f}, \nabla\nabla f\right\rangle\\
&- 2V|df|^2|dV|^2 - 2V^2\nabla\nabla f\nabla f dV - V^2|df|^2\nabla\nabla V\\
&+ 2V\left\langle df, dV\right\rangle df dV + V^2\nabla\nabla f dV df + V^2\left\langle df, \nabla\nabla V\right\rangle df\\
&+ V^2\left\langle df, dV\right\rangle\nabla\nabla f,\\
=&3V^2\Delta f\left\langle df, dV\right\rangle + V^3\left\langle d\Delta f, df\right\rangle + V^3\left(\Delta f\right)^2\\
&- 3V^2\left\langle \text{Hess f}, df\otimes dV\right\rangle - V^3\left\langle div^b\text{ Hess f}, df\right\rangle - V^3|\text{Hess f}|^2\\
&- 2V|df|^2|dV|^2 - 2V^2\left\langle \text{Hess f}, df\otimes dV\right\rangle - V^2\left\langle df\otimes df, \text{Hess V}\right\rangle\\
&+ 2V\left\langle df, dV\right\rangle^2 + V^2\left\langle \text{Hess f}, dV\otimes df\right\rangle + V^2\left\langle df\otimes df, \text{Hess V}\right\rangle\\
&+ V^2\left\langle df, dV\right\rangle\Delta f,\\
=&\underbrace{V^3\left[\left(\Delta f\right)^2 - |\text{Hess f}|^2 + \left\langle d\Delta f, df\right\rangle - \left\langle div^b\text{ Hess f}, df\right\rangle\right]}_{\textit{Distribute } V^3}\\
&- 4V^2\left\langle\text{ Hess f}, df\otimes dV\right\rangle + 4V^2\left\langle df, dV\right\rangle\Delta f\\
&+ 2V\left\langle df, dV\right\rangle^2 - 2V|df|^2|dV|^2.
\end{aligned}$$

Rewriting $\left\langle d\Delta f, df\right\rangle$

$$\left\langle d\Delta f, df\right\rangle = \left\langle \text{d Hess f}, df\right\rangle = \left\langle div^b\text{ Hess f}, df\right\rangle,$$

we see that

$$\begin{aligned}
=&V^3\left[\left(\Delta f\right)^2 - |\text{Hess f}|^2\right] - 4V^2\left\langle \text{Hess f}, df\otimes dV\right\rangle\\
&+ 4V^2\left\langle df, dV\right\rangle\Delta f + 2V\left\langle df, dV\right\rangle^2 - 2V|df|^2|dV|^2.
\end{aligned}$$

Looking at

$$\left\langle d\left(\frac{1}{1+V^2|df|^2}\right), \text{V } div^b e\ - \text{V d } tr^b e - e\left(\nabla V,\cdot\right) + \left(tr^b e\right) dV\right\rangle,$$

substituting in (5.21) (5.22) taking the derivative of $d\left(\frac{1}{1+V^2|df|^2}\right)$

$$\left\langle d\left(\frac{1}{1+V^2|df|^2}\right), \text{V } div^b e \ - \text{V d } tr^b e - e\left(\nabla V, \cdot\right) + \left(tr^b e\right) dV \right\rangle$$
$$= \left\langle \frac{-2V|df|^2 dV - 2V^2 \left\langle \text{Hess f}, df \otimes \cdot \right\rangle}{\left(1+V^2|df|^2\right)^2}, V^3 \nabla f df \ - V^3 \left\langle \text{Hess f}, df \otimes \cdot \right\rangle - V^2|df|^2 dV + V^2 \left\langle df, dV \right\rangle df \right\rangle,$$

distribute $V|df|^2 dV - V^2 \left\langle \text{Hess f}, df \otimes \cdot \right\rangle$

$$\left\langle d\left(\frac{1}{1+V^2|df|^2}\right), \text{V } div^b e \ - \text{V d } tr^b e - e\left(\nabla V, \cdot\right) + \left(tr^b e\right) dV \right\rangle$$
$$= \frac{-2}{\left(1+V^2|df|^2\right)^2} \left[V^4 \nabla f \left\langle df, dV \right\rangle - 2|df|^2 V^4 \left\langle \text{Hess f}, df \otimes dV \right\rangle \right.$$
$$- V^3|df|^4|dV|^2 + V^3|df|^2 \left\langle df, dV \right\rangle^2 + V^5 \nabla f \left\langle \text{Hess f}, df \otimes df \right\rangle$$
$$\left. -V^5 |\left\langle \text{Hess f}, df \otimes \cdot \right\rangle|^2 \ + V^4 \left\langle df, dV \right\rangle \left\langle \text{Hess f}, df \otimes df \right\rangle \right].$$

Now that we know the components from the divergence's chain rule, we can substitute them as follow

$$\text{div}^b \left[\frac{1}{1+V^2|df|^2} \left(V \ \text{div}^b e \ - \text{V d } tr^b e - e\left(\nabla V, \cdot\right) \ + \left(tr^b e\right) dV\right)\right]$$
$$= \frac{1}{1+V^2|df|^2} \left[V^3 \left(\left(\Delta f\right)^2 - |\text{Hess f}|^2\right) - 4V^2 \left\langle \text{Hess f}, df \otimes dV \right\rangle \right.$$
$$\left. +4V^2 \left\langle df, dV \right\rangle \Delta f + 2V \left\langle df, dV \right\rangle^2 - 2V|df|^2|dV|^2 \right]$$
$$- \frac{2}{\left(1+V^2|df|^2\right)}^2 \left[V^4 \Delta f \left\langle df, dV \right\rangle - |df|^2 V^4 \left\langle \text{Hess f}, df \otimes dV \right\rangle - V^3|df|^4|dV|^2 \right.$$
$$+ V^3|df|^2 \left\langle df, dV \right\rangle^2 + V^5 \Delta f \left\langle \text{Hess f}, df \otimes df \right\rangle$$
$$\left. - V^5 |\left\langle \text{Hess f}, df \otimes \cdot \right\rangle|^2 \ + V^4 \left\langle df, dV \right\rangle \left\langle \text{Hess f}, df \otimes df \right\rangle \right].$$

Factoring out $\frac{1}{1+V^2|df|^2}$, we ultimately see (5.5) but with a multiple of V

$$\text{div}^b \left[\frac{1}{1+V^2|df|^2} \left(V \ \text{div}^b e \ - \text{V d } tr^b e - e\left(\nabla V, \cdot\right) \ + \left(tr^b e\right) dV\right)\right]$$
$$= \frac{1}{1+V^2|df|^2} \left[V^3 \left(\left(\Delta f\right)^2 - |\text{Hess f}|^2\right) - \frac{2}{1+V^2|df|^2} \left(V^5 \Delta f \left\langle \text{Hess f}, df \otimes df \right\rangle\right) \right.$$
$$\left. -V^5 |\left\langle \text{Hess f}, df \otimes \cdot \right\rangle|^2\right)$$
$$- \frac{4V^2}{1+V^2|df|^2} \left\langle \text{Hess f}, df \otimes dV \right\rangle + \frac{2V}{1+V^2|df|^2} \left(\left\langle df, dV \right\rangle^2 - |df|^2|dV|^2\right)$$
$$\left. - \frac{2V^4}{1+V^2|df|^2} \left\langle df, dV \right\rangle \left\langle \text{Hess f}, df \otimes df \right\rangle + \left(2 + \frac{1}{1+V^2|df|^2}\right) \Delta f |df|^2 \left\langle df, dV \right\rangle \right],$$
$$= V\left(H^2 - |S|^2\right).$$

Thus we can conclude

(5.23)

$$V\left[R_g + n\left(n-1\right)\right] = V(H^2 - |S|^2) = \text{div}^b \left[\frac{1}{1+V^2|df|^2} \left(V \ \text{div}\ ^b e - \text{V d } tr^b e - e\left(\nabla V, \cdot\right) + \left(tr^b e\right) dV\right)\right].$$

$\square$

Proof of Theorem 5.1. Let ν denote the outgoing unit normal to $\partial\Omega$ and let $\nu_r = \partial_r$ be the normal to the spheres of constant r. By the Gauss equation (3.17) and Lemma 5.5, we have

$$V\left[R\left(g\right)+n\left(n-1\right)\right]=div^{b}\left[\frac{1}{1+V^{2}|df|^{2}}\left(\mathrm{V}\ div^{b}e-\mathrm{V}\ \mathrm{d}\ \mathrm{tr}^{b}e-e\left(\nabla V,\cdot\right)+\left(tr^{b}e\right)dV\right)\right]. \tag{5.24}$$

Integrating (5.24) over an outer domain and using the divergence theorem

$$\begin{aligned}
&\int_{\mathbb{H}^n} V\frac{\left(R_g+n(n-1)\right)}{\sqrt{1+V^2df^2}}dVol_g\\
&=\int_{H^n} V\left(R_g+n\left(n-1\right)\right)dVol_b,\\
&=\lim_{r\to\infty}\int_{B_r(o)} V(R_g+n(n-1))dVol_b,\\
&=\lim_{r\to\infty}\int_{B_r(o)} div^b\left[\frac{1}{1+V^2|df|^2}\left(\mathrm{V}\ div^be-\mathrm{V}\ \mathrm{d}\ \mathrm{tr}^be-e\left(\nabla V,\cdot\right)+\left(tr^be\right)dV\right)\right],\\
&=\lim_{r\to\infty}\int_{Sr(0)}\frac{1}{1+V^2|df|^2}\left(\mathrm{V}\ div^b\ \mathrm{e}-\mathrm{V}\ \mathrm{d}\ tr^b\ \mathrm{e}-e(\nabla V,\cdot)+(tr^b\ \mathrm{e})dV\right)(\nu_r)dSr,\\
&=mass_{AH}\left(\Sigma,g\right).
\end{aligned}$$

Therefore, if $R(g)\geq -n(n-1)$, the total mass is non-negative. $\square$

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