"Renormalization-group treatment of the long-ranged one-dimensional Ising model with random fields", Physical Review B (1987)

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Overall Point & Relavance

By following steps laid out in the paper, we can calculate the recursion relations for renormalized parameters like J (interaction energy), μ (chemical potential), h (uniform external field), and h_R (width of random field's distribution, i.e., random field strength). Such recursion relations allow us to study a system's (e.g., 1D Ising Model) behavior at different length scales and have a better understanding of the critical behavior and phase transitions in the presence of the random field for said system. Also, we are shown that initial Gaussian distributions may not hold under a renormalization-group procedure.

1D Ising Hamiltonian with Random Field

$$
\begin{aligned}
H = &- \sum_{i \neq j} \frac{J \; n_i n_j}{\sigma(1-\sigma)} \left(\left| \frac{r_i - r_j}{\tau} \right|^{1-\sigma} - 1 \right) + 2n\mu \\
&- h \sum_{i=0}^{2n} n_i \left(\frac{r_{i+1} - r_i}{\tau} \right) - \sum_{i=0}^{2n} n_i \int_{r_i}^{r_{i+1}} \frac{dx}{\tau} h(x)
\end{aligned}
\tag{1}
$$

Conditions for Satisfying Gaussian Distribution, consequential Hamiltonian

$$\overline{h(x)h(x')} = \delta\left(\frac{x-x'}{\tau}\right)h_R^2 \tag{2}$$

$$H_{eff}(r_1, \ldots, r_{2n}, \tau + d\tau) = H(r_i, \ldots, r_{2n}) + \left(\Delta E + 2n_i \int_r^{r'} \frac{dx}{\tau} h(x)\right)\theta \left(-\Delta E - 2n_i \int_r^{r'} \frac{dx}{\tau} h(x)\right) \tag{3}$$

Defining Previous Terms

$$\Delta E \equiv 2\mu + 2hn_i - \sum_{j=1}^{2n} \frac{2J\ n_i n_j}{\sigma(1-\sigma)} \left(\left| \frac{r_j - r'}{\tau} \right|^{1-\sigma} - \left| \frac{r_j - r}{\tau} \right|^{1-\sigma} \right) \quad (4)$$

$$\theta(-\Delta E - 2n_i h_r) = \theta(-2\mu - 2n_i h_r) + (2\mu - \Delta E)\delta(-2\mu - 2n_i h_r) \quad (5)$$

Obtaining recursion relations for J, μ, and h

(1)Average the effective Hamiltonian over the random field: By averaging the effective Hamiltonian, we can take into account the influence of the random field on the system's behavior. This step allows us to obtain an effective description of the system that captures the essential features of the disordered system.
(2)Absorb the term proportional to $d\tau = d\tau/\tau$: This term is incorporated into the first term of the effective Hamiltonian, which is the average of the Hamiltonian of the pure model (i.e., the model without the random field). By doing so, we can effectively account for the renormalization of the system as length scale is changed.

Cont.

(3)Integrate over the position and length of the small domain: The averaging process includes integration over the position and length of the small domain, which accounts for the influence of the local interactions and randomness on the system's behavior at the larger length scale.
(4) Obtain a Hamiltonian with renormalized parameters J, μ, and h: After performing the steps mentioned above, we can obtain a Hamiltonian that has the same form as the original Hamiltonian but with renormalized parameters. These renormalized parameters describe the system's behavior at the larger length scale.
(5) Drop the constant shift in the ground-state energy: Since the focus is on the critical behavior and phase transitions, which are not affected by a constant shift in the ground-state energy, this term can be disregarded in the analysis.

RG Recursion Results (1/2)

$l = \ln L$ where L is the linear dimension of the 1D system.

$$\frac{\partial J}{\partial l} = J(1 - \sigma - 4y^2), \ \frac{\partial h}{\partial l} = h(1 - 2y^2), \ \frac{\partial \mu}{\partial l} = \frac{J}{\sigma}$$

Obtaining recursion relation for h_R

(1)Antisymmetrize the unaveraged expression (4) with respect to the sign of the spins in the background domain containing the proposed small block. This antisymmetrization process is crucial for determining how the random field interacts with the spins in the system, which can influence the system's behavior at the larger length scale.
(2)Subtract off the mean: By subtracting the mean of the renormalized random field, we can isolate the fluctuations in the random field at the new length scale. These fluctuations are the key contributors to the randomness in the system at the larger scale.
(3)Calculate the variance of the field: The variance of the random field at the new length scale quantifies the strength of the randomness in the system. By calculating the variance, we can obtain the recursion relation for the renormalized random field.

Cont.

leading to the following

$$\int_a^b h'(x)\frac{dx}{\tau + d\tau} = \int_a^b h(x)\frac{dx}{\tau} - dl\int_a^b \frac{dr}{\tau}[(h_r - \mu - \Delta/2)\theta(-h_r - \mu) + (h_r - \mu - \Delta/2)\theta(h_r - \mu) + \Delta\overline{\theta(h_r - \mu)}] \quad (6)$$

where $\Delta \equiv 2\mu - \Delta E$ and $h_r = \int_r^{r'} (dx/\tau)h(x)$

RG Recursion Results (2/2)

$$\frac{\partial h_R}{\partial l} = h_R(\frac{1}{2} - 2y^2) \tag{7}$$

Fixed Point for Non-Gaussian Distribution Demonstration

$\omega \equiv h_R/J$ and $v \equiv h_R/\mu$, then from previous RG recursion results see that

$$\begin{aligned} \frac{\partial \omega}{\partial l} &= \omega(\sigma - \frac{1}{2} + 2y^2), \\ \frac{\partial v}{\partial l} &= v\left(\frac{1}{2} - \frac{v}{\sigma\omega} - 2y^2\right) \end{aligned} \tag{8}$$

where there is a fixed point at $2y^{*2} = \epsilon = \dfrac{1}{2} - \sigma$, fixing $\omega = 4v^*$.

Non-Gaussian Demonstration

The recursion relation for cumulants of this RG-generated system is

$$\frac{\partial \hat{C}_{2n}}{\partial l} = -\hat{C}_{2n}(n-1) - 4ny^2x^{2n-2}(1 + \hat{C}_4 x^4/4! + \hat{C}_6 x^6/6! \dots) \quad (9)$$

where $\hat{C}_n = -\frac{C_n}{h_R^n} = \frac{C_n}{C_2^{n/2}}$.

The central limit theorem tells us that

$$\frac{\partial \hat{C}_{2n}}{\partial l} \approx \hat{C}_{2n}(1-n) \quad (10)$$

but plugging into (8) a large x and n greater than 2 with a fixed ϵ, the higher-order cumulants are no longer ignorable as they're very large, thus a non-Gaussian distribution. Why n has to be greater than 2 is evident with (9) rewritten as a matrix equation

$$\hat{C}_n^* \simeq \frac{4}{n-2} y^2 x^{n-2} \quad (11)$$