Grandscale Review of Renormalization Group Theory

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Introduction

- Roots of RG theory are in critical phenomena
- Some people may ask how much is this instrumental, as opposed to how much far it is crucial? (i.e., how much of it is physically real, vs a neat mathematical tool)
- Statistical Mechanics is not a theory that is reduced, and is not directly reducible to lower levels without new postulates -> most theorems have been proved in full figure
- Will view RG as a tool or computational device… and how it fits into QFT and other areas of physics

Whence came renormalization group theory?

- Paint RG theory in the context of critical exponents and scaling factors
  - Cyril Domb and his group at UCL, Pokrovski and Patashinskii used field-theoretic perspectives
- Full RG concept brought about by Kenneth G. Wilson, also added to be Kadanoff and Wegner through their use of marginal operators
- However it's Landau's use of effective order parameters in field theories that is crucial
  - Use knowledge of microscopic behaviors/symmetries to see how larger aspects of the system behave
- Order parameter allows us to gain knowledge into the intermediate range ($10^{-6.5}$-$10^{-3.5}$cm) between atomic/nuclei scales (10^{-13}-10^{-8}cm) and macroscopic objects
- Wilson's work gave new meaning to coarse-grained hamiltonians
  - LGW-Hamiltonians are true but renormalized and have microscopic degrees of freedom integrated out

Whence came renormalization group theory? (cont.)

- Start of with LGW-Hamiltonian, then bring in statistical mechanics to understand macroscopic behavior
- 1944, Onsanger computed the partition function and thermodynamic properties of a fluid/simplest ferromagnet (i.e., the ising model)
  - Critical singularities disagreed with those expected in Landau Theory)
- This challenge, coupled with experimental evidence backing these predictions, gave rise to the idea of critical exponents… and ultimately influenced Ken Wilson to apply these ideas to quantum field theories
  - Mathematical analog with LGW-Hamiltonians in statistical mechanics and Feynman Path integrals in QFT
  - 1971: Wilson (and Wegner) revealed the beauty and effectiveness of RNGT
- Our understanding of anamlous (non-Landau) type transitions were greatly enhanced
- Some paradoxes have emerged, Arthur Wigmann: “Maybe we should go back and think more carefully about what we are actually doing in implementing theoretical ideas?”

Where stands the Renormalization Group?

- Mischaracterizations of RG by Itzykson, Drouffee and Benfatto, Gallavoti published misleading works, namely *Statistical Field theory* and *Renormalization Group* by claiming RGT is essentially a second order (lowest-order theory)
  - Implied also that QFT is necessary for RG, but disregarded monte carlo results and other uses of RG in real space (functional RG)
- Examples of problems treated by RG theory include: Kolmogorov-Arnold-Moser theory of hamiltonian stability, universality of critical points in statistical mechanics, Kondo problem for magnetic impurities in nonmagnetic materials, and many more
  - When Wilson solved the Kondo problem with RG and numerical techniques he wasn’t given much praise, even though it was a HUGE deal

Where stands the Renormalization Group? (cont.)

- Wilson explains that Feynman diagrams “completely hide the physics of many scales”
- Field theoretic techniques only work when the underlying physics is well understood, but RG can give insight into problems that are not well understood
- Wilson divides RG theory into 4 parts:
  - Theory of fixed points and linear/non-linear behavior near them
  - Field theoretic approach of RG for critical phenomena with small ε expansion
  - QFT methods -> Callan-Symanzik equations, Gell-Man-Low RG theory
  - Non-field theory RG transformations that are solved numerically

Exponents, anomalous dimensions, scale invariance and scale dependence

- Epitome of the success of RG theory is "It has to be stressed that the possibility of nonzero criticality is its most important achievement"
- Consider a microscopic variable $\Psi(r)$
  - In a magnet, this would be local magnetization M(r), or spin vector S(r)
  - in QFT these local variables are basic quantum fields, M(r), S(r) would become operators
- Observing the two point correlation function $G(r) = \langle\Psi(0)|\Psi(r)\rangle$
  - Provides measure of how much microscopic fluctuations at origin affect those at distance r=|r| away
  - Near critical point a strong "ordering" influence or correlation spreads out -> at critical point we find power-law decay correlation function. I.e. $G_c(r) \sim D/r^{d-2+\eta}$ ($d-2+\eta$ is critical exponent) ($r \gg a$)

Exponents, anomalous dimensions, scale invariance and scale dependence (cont.)

- η vanishes in Landau-Ginzberg or van der Waal’s (QFT -> massless particle)
  - In these theories basic functions are assumed to be analytic, non-singular and smooth to expand in positive powers in taylore series near critical point
- η implies that the fluctuations are either 0 or play a role only at much smaller scales -> incorporate into effective /renormalized parameters (masses, coupling constants, etc.)
- Power law dependence -> no length scale -> scale invariance: r’=br, order parameter by b^w
- $G'_c(br) \sim D/r^{d-2+\eta}$ -> $b^w D/(b*r)^{d-2+\eta}$, $w = ½(d-2+\eta)$-> same correlation function (i.e. invariance).
- Scale invariance means classical theories should be suspect near criticality (η isn’t zero)
  - RGT -> get anomalous values of η
- $\eta=0$, $w=½(d-2)$. For d=3, η is small and we can renormalize the amplitude $D'(R) = D/R^\eta$, $G'_c(r)=D'/r^{d-2}$
- Small η, D’ dies slowly on the scale of R. In QFT, logR dependence on normalization parameter, variation scale still weaker than when η isn’t zero

Challenges Posed by Critical Phenomena

- 1869 Andrews reported carbon dioxide coexisting between liquid and gas phases at a meniscus in a glass tube at room temperature. Raise the temp to T_c=31C,liquid becomes gas.
- Critical densities and concentrations “non-universal parameters” that reflect the microscopic physics of the system below size a
- No real physical symmetry between coexisting liquid and gas, just a dense and less dense state

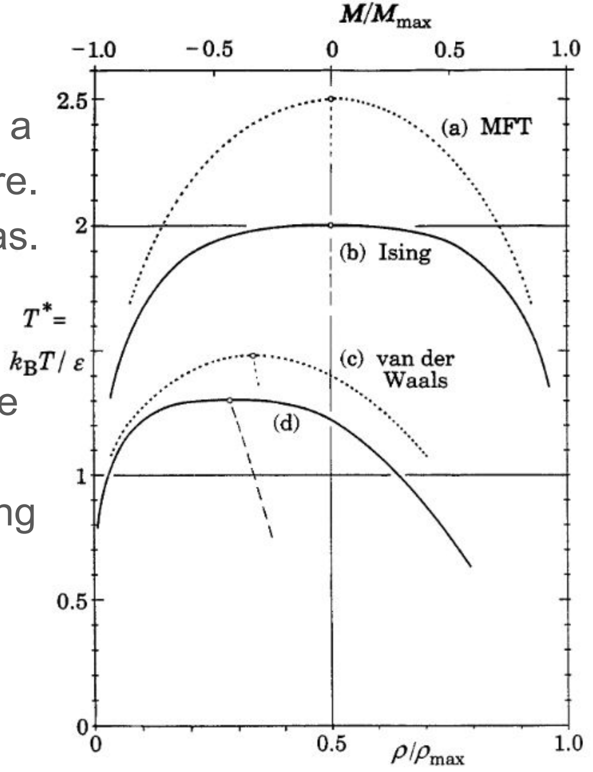


FIG. 1. Temperature variation of gas-liquid coexistence curves (temperature, T, *versus* density, ρ) and corresponding spontaneous magnetization plots (magnetization, M, *versus* T). The solid curves, (b) and (d), represent (semiquantitatively) observation and modern theory, while the dotted curves (a) and (c) illustrate the corresponding “classical” predictions (mean-field theory and van der Waals approximation). These latter plots are parabolic through the critical points (small open circles) instead of obeying a power law with the universal exponent $\beta \simeq 0.325$: see Eqs. (9) and (11). The energy scale ε, and the maximal density and magnetization, ρ_{max} and M_{max}, are nonuniversal parameters particular to each physical system; they vary widely in magnitude.

Challenges Posed by Critical Phenomena

- Asymptotic symmetry: $t = (T-T_c)/T_c$ as $T \to 0_-$. The fluid builds itself a mirror symmetry on the opposite side of the meniscus as T approaches T_c. $\Delta\rho = B|t|^\beta$, Critical exponent $\beta=0.325$ vs ½ expected classically (⅛ for square ising model, confirmed experimentally in 1984)
- $\beta=0.325$ also applies for certain magnetic materials. In these materials, below T_c can have spontaneous magnetization.
  - Global shape of spontaneous magnetization doesn't resemble normal coexistence curve
  - $M_0(T) \sim B|t|^\beta$ as $t \to 0_-$
- $C(T) = A_{\pm}/|t|^\alpha$ as $t \to 0_{+/-}$ and $A_+/A_- = 0.52$ universally, in 2D it's 1 and we have log|t| behavior
- Other central quantities: Divergent thermal compressibility κ and divergent correlation length ξ -range of influence or of correlation
  - Long range limits: $\chi(T) = C_{\pm}/|t|^\gamma$, $\xi = \xi_0^{+/-}/|t|^\nu$, $t \to 0$ in 3d, $\gamma=1.24$, $\nu=0.63$
- Early success of RG theory helped to understand universality classes

Exponent Relations, Scaling and Irrelevance

- Existence of universally scaling critical exponents was accepted in the 1960s
  - Now we have exponent relations, algebraic relations that are satisfied independent of class
  - $\gamma=(2-\eta)\nu$, $\alpha+2\beta+\gamma=2$. Hold for d=2, accurate to d=3 (classical d>4)
- Onsanger's solution to ising model: Correlation function scales as correlation length in all critical regions for all lengths greater than lattice spacing

$$G(T;\mathbf{r}) \approx \frac{D}{r^{d-2+\eta}} \mathcal{G}\left(\frac{r}{\xi(T)}\right)$$

- Equation of State
  - Classically it can be found using taylor expansions in $(T-T_c)$, etc.
  - This enforces classical exponents
- Widom's scaling hypothesis - minimize number of critical exponents
  - α isi from specific heat, $\Delta=\beta+\gamma$ (how h scales with T)
  - Wisdom: Classical obeys scattering: $\alpha=0$, $\Delta=1.5$, $\varphi=-0.5$

$$f_s(t,h,g) \approx |t|^{2-\alpha}\mathcal{F}\left(\frac{h}{|t|^{\Delta}}, \frac{g}{|t|^{\phi}}\right)$$

Exponent Relations, Scaling and Irrelevance (cont.)

- Φ, $z = g/|t|^{\varphi}$ not present in classical model. In RG we have a spectrum of Φ values
    - Spectrum is overlook as $-\Phi>0$, or $\Phi<0$, so the higher order coupling constants g are irrelevant and die off in RG treatment
    - T_c approached, same function F for all g -> *universality*
    - All of these systems will exhibit the same critical behavior determined by free energy function $\mathcal{F}$
- RGT implies scaling! $f_s(t,h,g) \approx |t|^{2-\alpha} \mathcal{F}\left(\frac{h}{|t|^{\Delta}}, \frac{g}{|t|^{\phi}}\right)$
    - Implies exponents like α,β,ɣ that are obtained through thermodynamic relations
    - Δ, ɑ determine all other leading exponents, so we can predict other exponents too
    - Fix Pressure (or g)-> obtain E.O.S./data with respect to two variables (say T,H)
        - Display as isotherms near Tc and plot scaled f or M vs scaled field h, data collapses to single curve $\mathcal{F}$
- Collapse is universal since the free energy density is as well
    - Vortex glass transition in high temp YBCO

Relevance, Crossover, Marginality

- Scaling behavior is valid when (T-Tc) is small, external field is small, and microscopic cutoff distance is less than what we're interested in
- If P∝g, $z=g/|t|^{\varphi}$ and becomes small as t->0 -> expand around scaling Function F(y,z) in powers of z
    - Spontaneous magnetization should then become
    - Above θ=-φ, so φ>0 yields a less interesting F(y,z) behavior $M_0(T)=B|t|^{\beta}(1+b_{\theta}|t|^{\theta}+b_1 t+\cdots),$
    - P is a relative perturbation, causing 1) critical point destruction or 2) new universality class and scaling function can be revealed with new critical exponents
- Ferromagnetic short range and long range dipole-dipole interactions solved with RG theory
    - New exponents obtained numerically for iron, nickel, are so close to short-range exponents that it almost doesn't matter
- d=3 Ising-type ferromagnets have dipolar couplings as marginal variables -> log|T-Tc| behavior.
    - d=2, marginal behavior in critical exponents…mention of more diverse applications in CM physics than QFT

The Task for Renormalization Group Theory

- RG theory wishes to:
  - i) should explain ubiquity of power law at and near critical point
    - Flow in some space of Hamiltonians H (or coupling constants)
    - The critical point is a fixed point of that flow
    - Flow operator (RG transformation R) is linearized about fixed point
    - Find R such that you can break the operator into eigenvalues and use that to find flow parameters and power laws☆☆☆
  - ii) obtain α,□,ɣ,δ,v and w; clarify why and how classical values are wrong
  - iv) correction-to-scaling exponent θ (1/φ) (and higher order)
  - v) crossover events, the universality of non-trivial exponents, and a derivation of scaling
  - vi)Handle breakdown of universality and exotic temperature dependencies

The Task for Renormalization Group Theory

- Consider spins s_x (spins, tensors, operators) at uniformly located space point x. With lattice spacing a, $V=Na^d$. The local magnetization and energy densities are
  - Have reduced hamiltonian $\bar{\mathcal{H}}[s; t, h, \cdots, h_j, \cdots] = -\mathcal{H}[\{s_{\mathbf{x}}\}; \cdots, h_j, \cdots]/k_B T,$
  - $\{S_x\}$ is the set of all spins, t, h, hj are the thermodynamic fields, we use t and h here
  - Wilson: a physical hamiltonian spanned by t&h is a subspace of the total hamiltonian space H $M_{\mathbf{x}} = \mu_B s_{\mathbf{x}}, \quad \mathcal{E}_{\mathbf{x}} = -\frac{1}{2} J \sum_{\delta} s_{\mathbf{x}} s_{\mathbf{x}+\delta}$ $f[\bar{\mathcal{H}}] \equiv f(t, h, \cdots h_j \cdots) = \lim_{N,V\to\infty} V^{-1} \log Z_N[\bar{\mathcal{H}}]$ $Z_N[\bar{\mathcal{H}}] = \mathrm{Tr}_N^s\{e^{\bar{\mathcal{H}}[s]}\}$
  -
  - Perform trace, take thermodynamic limit, get exact results (Onsanger in 1944)
  - While these results are known analytically, there's no real insight gained into why these are the results. RG can shed some light on the reasoning behind these exponents

Kadanoff's Scaling Picture

- In 1966 Kadanoff derived scaling by mapping a critical or near-critical system onto itself by a reduction of the numbers of degrees of freedom
  - Initially looked down upon, but the basics of his ideas were quite similar to those of Wilson's work
  - Consider ising model with spin values +/-1, and coupling constant J. Spontaneous magnetization can occur below a critical temperature Tc
  - Now divide block into L x L x … x L (L=b*a), b^d spins.
    - Each block B' has spin at center (s_x'), scale as x'=x/b, s_x' looks like s_x
  - Relating J'->J, h'->h, t'->t, we assume $b<\xi/a$ (ξ is correlation length)

Kadanoff's Scaling Picture (cont.)

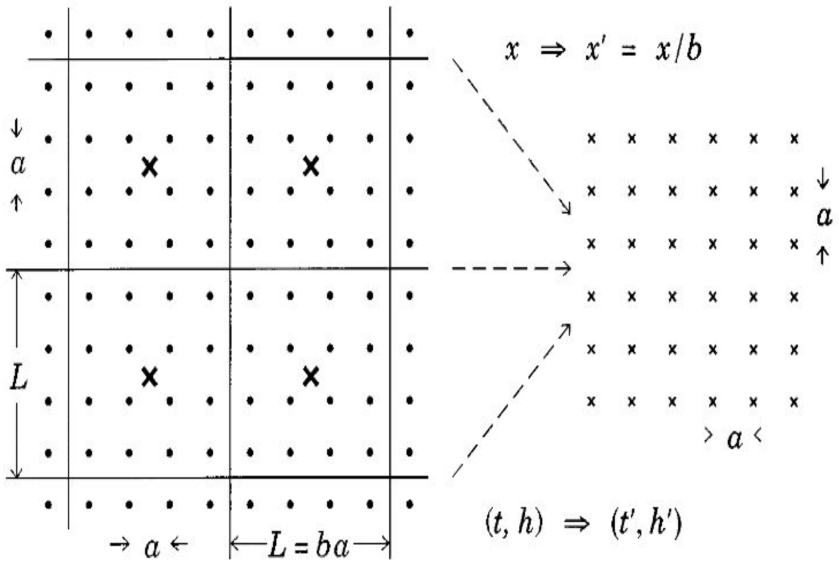
- 

FIG. 3. A lattice of spacing a of Ising spins $s_{\mathbf{x}} = \pm 1$ (in $d=2$ dimensions) marked by solid dots, divided up into Kadanoff blocks or cells of dimensions $(L=ba)\times(L=ba)$ each containing a block spin $s'_{\mathbf{x}'} = \pm 1$, indicated by a cross. After a rescaling, $\mathbf{x} \Rightarrow \mathbf{x}' = \mathbf{x}/b$, the lattice of block spins appears identical with the original lattice. However, one supposes that the temperature t, and magnetic field h, of the original lattice can be renormalized to yield appropriate values, t' and h', for the rescaled, block-spin lattice: see text. In this illustration the spatial rescaling factor is $b = 4$.

Kadanoff’s Scaling Picture (cont.)

- Scaling has the following affect:
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$$\bar{s}_{\mathbf{x}'} \equiv b^{-d} \sum_{\mathbf{x} \in \mathcal{B}_{\mathbf{x}'}} s_{\mathbf{x}} \cong \zeta(b) s'_{\mathbf{x}'}, \quad t' \approx \vartheta(b) t \quad \text{and} \quad h' \approx \zeta(b) h.$$

$$G(\mathbf{x};\, t,\, h) \equiv \langle s_{\mathbf{0}}\, s_{\mathbf{x}} \rangle \approx \zeta^2(b) G(\mathbf{x}';\, t', h')$$